ROMANIAN MATHEMATICAL MAGAZINE

S.2631 If $x, y, z, t > 0$, then:

$$(x^4+t^4)(y^4+t^4)(z^4+t^4) \geq \frac{t^8}{12}(x+y+z)^4$$

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We will use the following inequalities:

$$(a^2+t^2)(b^2+t^2)(c^2+t^2) \geq \frac{3}{4}t^4(a+b+c)^2 \quad (1)$$

(Alt's inequality, with equality if and only if $a = b = c = \frac{t}{\sqrt{2}}$)

and $3(a^2+b^2+c^2) \geq (a+b+c)^2 \quad (2)$. It follows that

$$(x^4+t^4)(y^4+t^4)(z^4+t^4) \geq \frac{3}{4}(t^2)^4(x^2+y^2+z^2)^2 \geq$$

$$\geq \frac{3t^8}{4}\left(\frac{(x+y+z)^2}{3}\right)^2 = \frac{t^8}{12}(x+y+z)^4.$$

Equality holds if and only if $x = y = z = \frac{t}{(2)^{1/4}}$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.