ROMANIAN MATHEMATICAL MAGAZINE

S.2632 If $a,b,c,x,y,z>0$, then:

$$\left(\frac{a^4}{x^2}+2\right)\left(\frac{b^4}{y^2}+2\right)\left(\frac{c^4}{z^2}+2\right)\ge\frac{3(a+b+c)^4}{(x+y+z)^2}.$$

Proposed by D.M.Bătinețu-Giurgiu, Elena Alexie – Romania

Solution by Titu Zvonaru-Romania

Applying Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$) and Bergström's inequality, it follows that

$$\left(\frac{a^4}{x^2}+2\right)\left(\frac{b^4}{y^2}+2\right)\left(\frac{c^4}{z^2}+2\right)\ge\frac{3}{4}(\sqrt{2})^4\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)^2\ge$$

$$\ge3\left(\frac{(a+b+c)^2}{x+y+z}\right)^2=\frac{3(a+b+c)^4}{(x+y+z)^2}.$$

Equality holds if and only if $a=b=c=\sqrt{x}=\sqrt{y}=\sqrt{z}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\ge\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\ge$$

$$\ge\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.