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S.2635 In any triangle ABC with the area F the following relationship holds:

$$\frac{(a^4+b^4)h_c}{a+b}+\frac{(b^4+c^4)h_a}{b+c}+\frac{(c^4+a^4)h_b}{c+a}\ge 8\sqrt{3}\cdot F^2$$

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We have $ah_a=bh_b=ch_c=2F$. Applying Bergström's inequality, the known inequality $a^2+b^2+c^2\ge ab+bc+ca$, and Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\ge 4\sqrt{3}\cdot F^2$, it follows that

$$\frac{(a^4+b^4)h_c}{a+b}+\frac{(b^4+c^4)h_a}{b+c}+\frac{(c^4+a^4)h_b}{c+a}=$$

$$=\frac{(a^4+b^4)ch_c}{ca+bc}+\frac{(b^4+c^4)ah_a}{ab+ca}+\frac{(c^4+a^4)bh_b}{bc+ab}=$$

$$=2F\left(\frac{a^4}{ca+bc}+\frac{b^4}{ab+ca}+\frac{c^4}{bc+ab}+\frac{a^4}{bc+ab}+\frac{b^4}{ca+bc}+\frac{c^4}{ab+ca}\right)\ge$$

$$=2F\left(\frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)}+\frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)}\right)=2F\cdot\frac{(a^2+b^2+c^2)(a^2+b^2+c^2)}{ab+bc+ca}\ge$$

$$\ge 2F(a^2+b^2+c^2)\ge 8\sqrt{3}\cdot F^2.$$

Equality holds if and only if the triangle ABC is equilateral.