S.2636 If $x,y,z>0$ and ABC is a triangle with semiperimeter s and the area F then:

$$\left(\frac{x^2a^4}{(y+z)^2}+R+s\right)\left(\frac{y^2b^4}{(z+x)^2}+R+s\right)\left(\frac{z^2c^4}{(x+y)^2}+R+s\right)\ge 72F^3$$

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Applying Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$) and Tsintsifas' inequality

$\frac{x}{y+x}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\ge 2\sqrt{3}F$, it follows that

$$\left(\frac{x^2a^4}{(y+z)^2}+R+s\right)\left(\frac{y^2b^4}{(z+x)^2}+R+s\right)\left(\frac{z^2c^4}{(x+y)^2}+R+s\right)\ge$$

$$\ge\frac{3}{4}\left(\sqrt{R+s}\right)^4\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)^2\ge\frac{3}{4}(R+s)^2\left(2\sqrt{3}F\right)^2=9(R+s)^2F^2\quad(1)$$

By $AM-GM$ and Euler's inequality $R\ge 2r$ we obtain $(R+s)^2\ge 4Rs\ge 8rs=8F\quad(2)$

Using (1) and (2) it results that

$$\left(\frac{x^2a^4}{(y+z)^2}+R+s\right)\left(\frac{y^2b^4}{(z+x)^2}+R+s\right)\left(\frac{z^2c^4}{(x+y)^2}+R+s\right)\ge 72F^3.$$

Equality holds if and only if $x=y=z, a=b=c$ and $\frac{a^2}{2}=\sqrt{\frac{R+s}{2}}$ that is $a^3=2\left(\frac{\sqrt{3}}{3}+\frac{3}{2}\right)$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\ge\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}\left((x+y)^2+t^2\right)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}\left(t(x+y)+tz\right)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.