ROMANIAN MATHEMATICAL MAGAZINE

S.2637 If $x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{x\cdot a^4}{y+z}+\frac{2(y+z)}{x}\right)\left(\frac{y\cdot b^4}{z+x}+\frac{2(z+x)}{y}\right)\left(\frac{z\cdot a^4}{x+y}+\frac{2(x+y)}{z}\right)\geq 288\cdot F^2$$

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Applying Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2) > \frac{3}{4}t^4(x+y+z)^2$

(with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$), Cesaro's inequality

$(x+y)(y+z)(z+x)\geq 8xyz$ and Tsintsifas' inequality

$\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\geq 2\sqrt{3}F$, it follows that

$$\left(\frac{x\cdot a^4}{y+z}+\frac{2(y+z)}{x}\right)\left(\frac{y\cdot b^4}{z+x}+\frac{2(z+x)}{y}\right)\left(\frac{z\cdot a^4}{x+y}+\frac{2(x+y)}{z}\right)=$$

$$=\frac{(x+y)(y+z)(z+x)}{xyz}\cdot\left(\frac{x^2\cdot a^4}{(y+z)^2}+2\right)\left(\frac{y^2\cdot b^4}{(z+x)^2}+2\right)\left(\frac{z^2\cdot a^4}{(x+y)^2}+2\right)\geq$$

$$\geq 8\cdot\frac{3}{4}\left(\sqrt{2}\right)^4\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)^2\geq 24\left(2\sqrt{3}F\right)^2=228\cdot F^2.$$

Equality holds if and only if $a=b=c, x=y=z, \frac{x}{y+z}a^2=1$,

that is $x=y=z, a=b=c=\sqrt{2}$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.