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SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE

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Abstract :
In this article, we establish new geometric inequalities in triangle involving Spieker′s Cevians.

We consider ΔABC with usual notations. Let p_a, p_a, p_a be the Spieker′s Cevians in ΔABC.

Lemma 1.

Given two points $\boldsymbol{P = (x_1, y_1, z_1)}$ and $\boldsymbol{Q = (x_2, y_2, z_2)}$ in normalized barycentric coordinates in the plane of the triangle $\boldsymbol{ABC}$. Then :

$$\boldsymbol{PQ^2 = -a^2(y_1 - y_2)(z_1 - z_2) - b^2(z_1 - z_2)(x_1 - x_2) - c^2(x_1 - x_2)(y_1 - y_2).}$$

The above formulae is well known (see [1, pp. 11]).

Note that the barycentric coordinates of A and S_p are

$$\mathrm{A} = (1 : 0 : 0) \text{ and } S_p = (b + c : c + a : a + b).$$

If D is the point of intersection of the lines AS_p and BC, then we have

$D = (0 : c + a : a + b)$, and by using the Lemma 1, we have

$$p_a{}^2 = DA^2 = -a^2 . \frac{c+a}{2a+b+c} . \frac{a+b}{2a+b+c} - b^2 . \frac{a+b}{2a+b+c} . (-1) - c^2(-1) . \frac{c+a}{2a+b+c}$$

$$\Rightarrow \ \boldsymbol{p_a{}^2 = \frac{b^3 + c^3 + a(b^2 + c^2)}{2a + b + c} - \frac{a^2(a+b)(a+c)}{(2a+b+c)^2}}. \qquad (\boldsymbol{1'})$$

Lemma 2. For any triangle $\boldsymbol{ABC}$, we have

$$\boldsymbol{p_a{}^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}}. \qquad (\boldsymbol{2'})$$

Proof. Using the identity $(1')$, we have

$$p_a{}^2 = \frac{(b+c+2a)(b+c)^2 + [3(b+c)+2a](b-c)^2}{4(2a+b+c)} - \frac{a^2[(2a+b+c)^2 - (b-c)^2]}{4(2a+b+c)^2}$$

$$= \frac{(b+c)^2 - a^2}{4} + \frac{(2a+b+c)[3(b+c)+2a] + a^2}{4(2a+b+c)^2} . (b-c)^2$$

$$= s(s-a) + \frac{(a+b+c)(5a+3b+3c)}{4(2a+b+c)^2}.(b-c)^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}.$$

Lemma 3. For any triangle ABC, we have

$$\boldsymbol{m_a \le p_a \le n_a}. \qquad (3')$$

Equality holds if and only if $b = c$.

Proof. We have the following known formulas

$$m_a{}^2 = s(s-a) + \frac{(b-c)^2}{4}, \qquad n_a{}^2 = s(s-a) + \frac{s(b-c)^2}{a}.$$

Since $\frac{s(3s+a)}{(2s+a)^2} = \frac{1}{4} + \frac{8s^2-a^2}{4(2s+a)^2} \ge \frac{1}{4}$ and $\frac{3s+a}{(2s+a)^2} =$

$= \frac{1}{a} - \frac{4s^2+sa}{a(2s+a)^2} \le \frac{1}{a}$, then by using (2),

we obtain : $m_a \le p_a \le n_a$.

Lemma 4. For any triangle ABC, we have

$$\boldsymbol{(2s+a)(2s+b)(2s+c) = 2s(9s^2+r^2+6Rr)}. \qquad (4')$$

$$\boldsymbol{\frac{1}{2s+a} + \frac{1}{2s+b} + \frac{1}{2s+c} = \frac{21s^2+r^2+4Rr}{2s(9s^2+r^2+6Rr)}}. \qquad (5')$$

$$\boldsymbol{\sum_{cyc}(2s+a){p_a}^2 = 2s.\frac{21s^4 - s^2(92Rr+30r^2) - 64R^2r^2 - 28Rr^3 - 3r^4}{9s^2+r^2+6Rr}}. \qquad (6')$$

$$\boldsymbol{\sum_{cyc}(2s+a)^2{p_a}^2 = 4s^2(3s^2-r^2-16Rr)}. \qquad (7')$$

$$\boldsymbol{\sum_{cyc}(2s+a)^3{p_a}^2 = 32s^3(s^2+r^2-6Rr)}. \qquad (8')$$

Proof. With known identities [2, pp. 52],

$$a+b+c = 2s, \qquad abc = 4Rsr$$

$$a^2+b^2+c^2 = 2(s^2-r^2-4Rr), \qquad ab+bc+ca = s^2+r^2+4Rr$$

$$a^3+b^3+c^3=2s(s^2-3r^2-6Rr),$$

$$a^4+b^4+c^4=2s^4-4(4Rr+3r^2)s^2+2(4R+r)^2r^2,$$

we obtain

$$(2s+a)(2s+b)(2s+c)=8s^3+4s^2(a+b+c)+2s(ab+bc+ca)+abc$$

$$=2s(9s^2+r^2+6Rr).$$

$$\frac{1}{2s+a}+\frac{1}{2s+b}+\frac{1}{2s+c}=\frac{12s^2+4s(a+b+c)+(ab+bc+ca)}{(2s+a)(2s+b)(2s+c)}$$
$$=\frac{21s^2+r^2+4Rr}{2s(9s^2+r^2+6Rr)}.$$

Now, since we have

$$a^2(a+b)(a+c)=a^2(2sa+bc)=a(2sa^2+4sRs)$$

$$=[(2s+a)-2s].2s[4s^2+2Rr-(2s+a)(2s-a)]$$

$$=2s(2s+a)(a^2+4s^2+2Rr-2sa)-8s^2(2s^2+Rr).$$

then, by using the identity $(1')$, we obtain

$$(2s+a){p_a}^2=b^3+c^3+a(b^2+c^2)-\frac{a^2(a+b)(a+c)}{2s+a}$$

$$=b^3+c^3+a(b^2+c^2)-2s(a^2+4s^2+2Rr-2sa)+\frac{8s^2(2s^2+Rr)}{2s+a}.$$

Adding this identity with its similar ones and using the identity $(5')$, we obtain

$$\sum_{cyc}(2s+a){p_a}^2=\sum a^3+\sum a^2.\sum a-2s\sum a^2-6s(4s^2+2Rr)+4s^2\sum a+$$

$$+8s^2(2s^2+Rr)\left(\frac{1}{2s+a}+\frac{1}{2s+b}+\frac{1}{2s+c}\right)$$

$$=2s.\frac{21s^4-s^2(92Rr+30r^2)-64R^2r^2-28Rr^3-3r^4}{9s^2+r^2+6Rr}.$$

Now, by using the identity $(2')$, we have

$$(2s+a)^2{p_a}^2=s(s-a)(2s+a)^2+s(3s+a)(b-c)^2$$

$$= 4s^4 - 2sa^3 + 3s^2(b^2 + c^2 - a^2 - 2bc) + s(a^2 + b^2 + c^2)a - 2sabc,$$

Adding this identity with its similar ones, we obtain

$$\sum_{cyc}(2s + a)^3 {p_a}^2 = 12s^4 - 2s\sum a^3 + 3s^2\sum(a^2 - 2bc) + s\sum a^2 . \sum a - 6sabc$$

$$= 4s^2(3s^2 - r^2 - 16Rr).$$

Also, by using the identity $(2')$, we have

$$(2s + a)^3 {p_a}^2 = s(s - a)(2s + a)^3 + s(3s + a)(2s + a)(b - c)^2$$

$$= 8s^5 + 2s(2s^3 - abc)a - 10s^2a^3 - 2sa^4 - 12s^3(a^2 + bc)$$
$$+ (6s^3 + 5s^2a + sa^2)(a^2 + b^2 + c^2) - 10s^2abc,$$

Adding this identity with its similar ones, we obtain

$$\sum_{cyc}(2s + a)^3 {p_a}^2$$
$$= 24s^5 + 2s(2s^3 - abc)\sum a - 10s^2\sum a^3 - 2s\sum a^4$$
$$- 12s^3\sum(a^2 + bc) + \sum a^2 . \left(18s^3 + 5s^2\sum a + s\sum a^2\right) - 30s^2abc$$
$$= 32s^3(s^2 + r^2 - 6Rr).$$

Theorem 1. For any triangle ABC, we have

$$\boldsymbol{p_a + p_b + p_c \leq \frac{14R - r}{3}}. \qquad \mathbf{(9')}$$

Equality holds if and only if the triangle ABC is equilateral.

Proof. By using the CBS inequality and the results $(5')$ and $(6')$, we have

$$(p_a + p_b + p_c)^2$$
$$\leq \left((2s + a){p_a}^2 + (2s + b){p_b}^2 + (2s + c){p_c}^2\right)\left(\frac{1}{2s + a} + \frac{1}{2s + b} + \frac{1}{2s + c}\right)$$

$$= \frac{[21s^4 - s^2(92Rr + 30r^2) - 64R^2r^2 - 28Rr^3 - 3r^4](21s^2 + r^2 + 4Rr)}{(9s^2 + r^2 + 6Rr)^2}$$

$$= (21s^2 + r^2 + 4Rr). f(s^2),$$

where $f(s^2)=\dfrac{21s^4-(92Rr+30r^2)s^2-64R^2r^2-28Rr^3-3r^4}{(9s^2+r^2+6Rr)^2}$.

We have $f'(s^2)=\dfrac{24(45Rr+13r^2)s^2+8(75R^2+29Rr+3r^2)r^2}{(9s^2+r^2+6Rr)^3}$
>0, then f is increasing,

and by using Gerretsen′s inequality $s^2\le 4R^2+4Rr+3r^2$, we obtain

$$(p_a+p_b+p_c)^2\le(21s^2+r^2+4Rr).f(s^2)$$
$$\le(84R^2+88Rr+64r^2).f(4R^2+4Rr+3r^2)$$

$$=\frac{16(441R^6+861R^5r+1132R^4r^2+805R^3r^3+524R^2r^4+212Rr^5+96r^6)}{(18R^2+21Rr+14r^2)^2}$$

$$=\left(\frac{14R-r}{3}\right)^2$$
$$-\frac{(R-2r)(15120R^4r+31608R^3r^2+36840R^2r^3+21121Rr^4+6814r^5)}{9(18R^2+21Rr+14r^2)^2}$$

$$\le\left(\frac{14R-r}{3}\right)^2,$$

the last line is true by Euler′s inequality

$R\ge 2r$, with equality if and only if ABC is equilateral.

This completes the proof of Theorem 1.

Theorem 2. For any triangle ABC, we have

$$\boldsymbol{p_ap_bp_c\le\frac{(8R-7r)s^2}{9}}. \qquad \mathbf{(10')}$$

Equality holds if and only if the triangle ABC is equilateral.

Proof. By using the AM − GM inequality, we have

$$(2s+a)^2{p_a}^2+(2s+b)^2{p_b}^2+(2s+c)^2{p_c}^2\ge 3\sqrt[3]{\left((2s+a)(2s+b)(2s+c)p_ap_bp_c\right)^2}.$$

Using the identities (4′) and (7′), we obtain

$$(p_ap_bp_c)^2\le\frac{[(2s+a)^2{p_a}^2+(2s+b)^2{p_b}^2+(2s+c)^2{p_c}^2]^3}{27[(2s+a)(2s+b)(2s+c)]^2}=\frac{[4s^2(3s^2-r^2-16Rr)]^3}{27[2s(9s^2+r^2+6Rr)]^2}$$

$$=\frac{16s^4(3s^2-r^2-16Rr)}{243}.\left(1-\frac{54Rr+4r^2}{9s^2+r^2+6Rr}\right)^2,$$

and by using Gerretsen′s inequality $s^2 \leq 4R^2+4Rr+3r^2$, we obtain

$$(p_ap_bp_c)^2 \leq \frac{16s^4(12R^2-4Rr+8r^2)}{243}.\left(1-\frac{54Rr+4r^2}{36R^2+42Rr+28r^2}\right)^2$$

$$=\frac{256(3R^2-Rr+2r^2)^3}{27(18R^2+21Rr+14r^2)^2}.s^4.$$

To complete the proof it is enough to prove that

$$768(3R^2-Rr+2r^2)^3 \leq (8R-7r)^2(18R^2+21Rr+14r^2)^2,$$

which is equivalent to

$$r(R-2r)(32832R^4+8964R^3r+15180R^2r^2-8903Rr^3-1730r^4) \geq 0,$$

which is true by Euler′s inequality $R \geq 2r$, with equality if and only if ABC is equilateral.

This completes the proof of Theorem 2.

Theorem 3. For any triangle ABC, we have

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c} \geq \frac{2}{R}. \qquad (11')$$

Equality holds if and only if the triangle ABC is equilateral.

Proof. By using Hölder′s inequality, we have

$$\left(\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}\right)^2 \geq \frac{[(2s+a)+(2s+b)+(2s+c)]^3}{(2s+a)^3{p_a}^2+(2s+b)^3{p_b}^2+(2s+c)^3{p_c}^2}$$

$$=\frac{(8s)^3}{32s^3(s^2+r^2-6Rr)}=\frac{16}{s^2+r^2-6Rr},$$

and by using Gerretsen′s inequality $s^2 \leq 4R^2+4Rr+3r^2$,

and Euler′s inequality $R \geq 2r$, we get

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c} \geq \frac{4}{\sqrt{4R^2-2r(R-2r)}} \geq \frac{2}{R},$$

with equality if and only if ABC is equilateral. This completes the proof of Theorem 3.

The main aim of the following part is to establish a geometric inequality involving Speiker's cev**ians angle bisectors and medians of a triangle, and applications of this inequality**.

Theorem 4. For any triangle ABC, we have

$$\boldsymbol{p_a w_a \le m_a{}^2}. \qquad (\mathbf{12'})$$

Equality holds if and only if $b = c$.

Proof. By the formulas for median and angle bisector of triangle ABC,

$$m_a{}^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) \text{ and } w_a = \frac{2\sqrt{bcs(s-a)}}{b+c}, \text{we can easily get}$$

$$m_a{}^2 = s(s-a) + \frac{(b-c)^2}{4} \text{ and } w_a{}^2 = s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2}.$$

and by using the identity $(2')$, we can get

$$p_a{}^2 = m_a{}^2 + \frac{(8s^2 - a^2)(b-c)^2}{4(2s+a)^2} \text{ and } w_a{}^2 = m_a{}^2 - \frac{(8s^2 - 8sa + a^2)(b-c)^2}{4(2s-a)^2}.$$

Based on these results, we have

$$p_a{}^2 w_a{}^2 = \left(m_a{}^2 + \frac{(8s^2 - a^2)(b-c)^2}{4(2s+a)^2}\right)\left(m_a{}^2 - \frac{(8s^2 - 8sa + a^2)(b-c)^2}{4(2s-a)^2}\right)$$

$$\le m_a{}^4 - m_a{}^2\left(\frac{8s^2 - 8sa + a^2}{(2s-a)^2} - \frac{8s^2 - a^2}{(2s+a)^2}\right)\frac{(b-c)^2}{4}$$

$$= m_a{}^4 - \frac{m_a{}^2[4sa(s-a)(4s+a) + a^4](b-c)^2}{2(4s^2 - a^2)^2} \le m_a{}^4,$$

the last line is true because $s > a$, with equality if and only if $b = c$.

This completes the proof of Lemma 5.

By Lemma 5 and some known identities a inequalities, we can get the following results

Corollary 1. For any triangle ABC, we have

$$p_aw_a + p_bw_b + p_cw_c \le \frac{3}{4}(s^2 - r^2 - 4Rr). \quad (13')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 2. For any triangle ABC, we have

$$p_aw_a + p_bw_b + p_cw_c \le \frac{3}{2}(2R^2 + r^2). \quad (14')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 3. For any triangle ABC, we have

$$ap_aw_a + bp_bw_b + cp_cw_c \le \frac{s}{2}(s^2 + 5r^2 + 2Rr). \quad (15')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 4. For any triangle ABC, we have

$$(b+c)p_aw_a + (c+a)p_bw_b + (a+b)p_cw_c \le \frac{s}{2}(5s^2 - 11r^2 - 26Rr). \quad (16')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 5. For any triangle ABC, we have

$$\sqrt{p_aw_a} + \sqrt{p_bw_b} + \sqrt{p_cw_c} \le m_a + m_b + m_c. \quad (17')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 6. For any triangle ABC, we have

$$\sqrt{p_aw_a} + \sqrt{p_bw_b} + \sqrt{p_cw_c} \le 4R + r. \quad (18')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 7. For any triangle ABC, we have

$$w_a + w_b + w_c \le \frac{{m_a}^2}{p_a} + \frac{{m_b}^2}{p_b} + \frac{{m_c}^2}{p_c} \le m_a + m_b + m_c. \quad (19')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 8. For any triangle ABC, we have

$$p_a + p_b + p_c \leq \frac{{m_a}^2}{w_a} + \frac{{m_b}^2}{w_b} + \frac{{m_c}^2}{w_c}. \qquad (20')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 9. For any triangle ABC, we have

$$\frac{p_a}{m_a} + \frac{p_b}{m_b} + \frac{p_c}{m_c} \leq \frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c}. \quad (\textit{Soumava Chakraborty}) \qquad (21')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 10. For any triangle ABC, we have

$$\frac{p_a w_a}{h_a m_a} + \frac{p_b w_b}{h_b m_b} + \frac{p_c w_c}{h_c m_c} \leq \frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq \frac{3R}{2r}. \quad (\textit{Soumava Chakraborty}) \qquad (22')$$

Equality holds if and only if the triangle ABC is equilateral.

Corollary 11. For any triangle ABC, we have

$$\frac{\sqrt{p_a w_a}}{h_a} + \frac{\sqrt{p_b w_b}}{h_b} + \frac{\sqrt{p_c w_c}}{h_c} \leq \frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq \frac{3R}{2r}. \qquad (23')$$

Equality holds if and only if the triangle ABC is equilateral.

In this part, we will use the Theorem 4. to obtain new results;

After summation we obtain $\sum p_a l_a \leq \sum m_a^2$

$$\sum m_a^2 = \frac{3}{4}(a^2+b^2+c^2) \to \frac{4}{3}\sum p_a l_a \leq a^2+b^2+c^2 \ (1)$$

We use well-known identity: $m_a^2 = r_b r_c + \frac{1}{4}(b-c)^2$ (and analogs) and we obtain:

$$\frac{1}{2}\,|b\text{-}c\,| \geq \sqrt{p_a l_a - r_b r_c} \text{ (and analogs) } (2)$$

We use $4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c$ (and analogs) [4] and obtain:

$$n_a^2 + g_a^2 \geq 4p_a l_a - 2r_b r_c \text{ (and analogs) } (3)$$

From (3) we obtain :

$$n_a + g_a \geq \sqrt{4p_a l_a + 2n_a g_a - 2r_b r_c} \text{ (and analogs) } (4)$$

But $n_a g_a \geq m_a l_a$ (and analogs) and from (4) we obtain:

$$\mathbf{n_a + g_a \geq \sqrt{4p_al_a + 2m_al_a - 2r_br_c}\ (and\ analogs)\ (5)}$$

But $(b-c)^2=(n_a-g_a)^2+2(n_ag_a-r_br_c)$ (and analogs) and (2) we obtain:

$$\mathbf{n_a - g_a \geq \sqrt{|4p_al_a - 2n_ag_a - 2r_br_c|}\ (and\ analogs)\ (6)}$$

From (4) and (6) after summation we obtain:

$$\mathbf{2n_a \geq \sqrt{4p_al_a + 2n_ag_a - 2r_br_c} + \sqrt{|4p_al_a - 2n_ag_a - 2r_br_c|}\ (and\ analogs)\ (7)}$$

$$\mathbf{4m_a^2 = 2(b^2+c^2) - a^2 \geq 4p_al_a \rightarrow b^2+c^2 \geq \frac{1}{2}(4p_al_a + a^2)\ (and\ analogs)\ (8)}$$

From (8) we obtain : $\frac{b}{c}+\frac{c}{b} \geq \frac{4p_al_a+a^2}{2bc}$ (and analogs)

But $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs)(Traian Lalescu)[5]

ω-Brocard angle in ΔABC, We obtain:

$$\mathbf{\frac{\sin(A+\omega)}{\sin\omega} \geq \frac{4p_al_a+a^2}{2bc}\ (and\ analogs)(9)}$$

From (9) we obtain:

$$\mathbf{\frac{1}{\sin\omega} \geq \frac{4p_al_a + a^2}{2bc}\ (and\ analogs)(10)}$$

Triangle ABC with sides a, b, c and triangle with sides m_a, m_b, m_c have same Brocard angle; [6] From $\frac{1}{\sin\omega} \geq \frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ and theorem above we obtain:

$$\frac{1}{\sin\omega} \geq \frac{m_b}{m_c}+\frac{m_c}{m_b}\ \text{(and analogs) and from (10) we obtain:}$$

$$\mathbf{\frac{1}{\sin\omega} \geq \frac{1}{2}\left(\frac{4p_al_a+a^2}{2bc}+\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)(and\ analogs)\ (11)}$$

$$\mathbf{\frac{1}{\sin\omega} \geq \frac{1}{2}\left(\frac{4p_al_a+a^2}{2bc}+\frac{m_b}{m_a}+\frac{m_a}{m_b}\right)(and\ analogs)\ (12)}$$

$$\mathbf{\frac{1}{\sin\omega} \geq \sqrt{\frac{4p_al_a+a^2}{2bc}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}(and\ analogs)\ (13)}$$

$$\mathbf{\frac{1}{\sin\omega} \geq \sqrt{\frac{4p_al_a+a^2}{2bc}\left(\frac{m_b}{m_a}+\frac{m_a}{m_b}\right)}(and\ analogs)\ (14)}$$

From $p_al_a \leq m_a^2$ (and analogs) and $\frac{m_a^2}{h_a^2}=1+\frac{(b^2-c^2)^2}{16S^2}$(and analogs) and

$2S=ah_a = bh_b = ch_c$, $b^2-c^2=(b+c)(b-c)$ we obtain:

$$\frac{b+c}{2a}|b-c|\geq\sqrt{p_a l_a - h_a^2} \text{ (and analogs) (15)}$$

From $p_a l_a \leq m_a^2$ (and analogs) and $\sum\frac{m_a^2}{h_a^2}=1+\frac{1}{2\sin^2\omega}$ we obtain:

$$1+\frac{1}{2\sin^2\omega}\geq\sum\frac{p_a l_a}{h_a^2} \text{ (16)}$$

From Catalan inequality:

$$a^2b\,(a-b)+b^2c\,(b-c)+c^2a\,(c-a)\geq 0$$

$$\to a^3b+b^3c+c^3a\geq a^2b^2+b^2c^2+c^2a^2 \to \frac{a^3b+b^3c+c^3a}{4S^2}\geq\frac{a^2b^2+b^2c^2+c^2a^2}{4S^2}=\frac{1}{\sin^2\omega}$$

$\to \frac{ab}{h_a^2}+\frac{bc}{h_b^2}+\frac{ca}{h_c^2}\geq\frac{1}{\sin^2\omega}$ and (16) we obtain:

$$1+\frac{1}{2}\left(\frac{ab}{h_a^2}+\frac{bc}{h_b^2}+\frac{ca}{h_c^2}\right)\geq\sum\frac{p_a l_a}{h_a^2} \text{(17)}$$

From $4m_a=2\sqrt{2(b^2+c^2)-a^2}$ (and analogs) and using AM-GM we obtain:

$4m_a\sqrt{a^2+b^2+c^2}\leq 2(b^2+c^2)-a^2+a^2+b^2+c^2=3(b^2+c^2)$ and

$m_a\geq\sqrt{p_a l_a}$(and analogs) we obtain:

$$\frac{4}{3}\sqrt{p_a l_a(a^2+b^2+c^2)}\leq b^2+c^2 \text{ (and analogs)(18)}$$

We know that $bc=2Rh_a$(and analogs) and from (18) we obtain:

$$2\frac{\sqrt{p_a l_a}}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R}\leq\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b} \text{ (19)}$$

From (19) we obtain:

$$2\max\left\{\frac{\sqrt{p_a l_a}}{h_a},\frac{\sqrt{p_b l_b}}{h_b},\frac{\sqrt{p_c l_c}}{h_c}\right\}\frac{\sqrt{a^2+b^2+c^2}}{3R}\leq\frac{1}{\sin\omega} \text{ (20)}$$

From (19) after summation we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\sum\frac{\sqrt{p_a l_a}}{h_a}\leq\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c} \text{ (21)}$$

From (21) and $2S=ah_a=bh_b=ch_c$, $\frac{b}{c}=\frac{h_c}{h_b}$ (and analogs) we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\sum\frac{\sqrt{p_a l_a}}{h_a}\leq\sum\frac{h_b+h_b}{h_a} \text{ (22)}$$

From (19) we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\sum\frac{\sqrt{p_a l_a}}{h_a}\le\sum\frac{\sin(A+\omega)}{\sin\omega}\ \textbf{(23)}$$

We proved $\frac{1}{\sin\omega}\ge\frac{m_b}{h_b}+\frac{m_c}{h_c}$ (and analogs) [7,8] and from $m_a\ge\sqrt{p_a l_a}$ (and analogs) we obtain:

$$\frac{1}{\sin\omega}\ge\frac{\sqrt{p_b l_b}}{h_b}+\frac{\sqrt{p_c l_c}}{h_c}\ \textbf{(and analogs) (24)}$$

From $l_a=\frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)}=\frac{2\sqrt{bc}}{b+c}\sqrt{r_b r_c}\to\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)=\frac{\sqrt{r_b r_c}}{l_a}$ (and analogs)

$4\frac{r_b r_c}{l_a^2}=2+\frac{b}{c}+\frac{c}{b}=2+\frac{\sin(A+\omega)}{\sin\omega}$ and using (9) we obtain:

$$4\frac{r_b r_c}{l_a^2}\ge 2+\frac{4p_a l_a+a^2}{2bc}\textbf{(and analogs) (25)}$$

From $m_a l_a\ge p(p-a)=r_b r_c$(and analogs)(Panaitopol) and (25) we obtain:

$$4\frac{m_a}{l_a}\ge 2+\frac{4p_a l_a+a^2}{2bc}\textbf{(and analogs) (26)}$$

$(a^2+b^2)(a^2+c^2)\ge(ab+ac)^2$(C.B.S inequality)

$m_b\ge\frac{a^2+c^2}{4R}$(and analogs) (Tereshin Inequality), $ac=2Rh_b$(and analogs)

After simple manipulations we obtain:

$2\sqrt{m_b m_c}\ge h_b+h_c$ (and analogs) and using $m_a\ge\sqrt{p_a l_a}$ (and analogs) we obtain after summation:

$$3\sqrt{m_b m_c}\ge h_b+h_c+\sqrt[4]{p_c l_c p_b l_b}\ \textbf{(and analogs) (27)}$$

From $n_a+g_a\ge 2m_a$(and analogs)[9]

$2m_a\ge 2\sqrt{p_a l_a}$ (and analogs) we obtain:

$$n_a+g_a\ge 2\sqrt{p_a l_a}\textbf{(and analogs)(28)}$$

From (28) and $n_a+g_a\ge 2\sqrt{n_a g_a}$(and analogs) after summation we obtain:

$$n_a+g_a\ge\sqrt{p_a l_a}+\sqrt{n_a g_a}\textbf{(and analogs) (29)}$$

From $\frac{1}{2}(|b-c|+|a-c|+|b-a|)=\max(a,b,c)-\min(a,b,c)$ and (2) we obtain:

$$\max(a,b,c)-\min(a,b,c)\ge\sum\sqrt{p_a l_a-r_b r_c}\textbf{(30)}$$

From $m_a l_a\ge p(p-a)=r_b r_c$(and analogs)(Panaitopol) and (2) we obtain:

$$\frac{1}{2}|b-c|\geq\sqrt{l_a(p_a-m_a)}\text{(and analogs) (31)}$$

From (31) after summation we obtain:

$$\max(a,b,c)-\min(a,b,c)\geq\sum\sqrt{l_a(p_a-m_a)}\text{(32)}$$

We use the identity: $\frac{n_a^2}{h_a^2}=1+\frac{(b-c)^2}{4r^2}$(and analogs) and (2) and obtain:

$$\frac{n_a}{h_a}\geq\frac{\sqrt{r^2+p_al_a-r_br_c}}{r}\text{(and analogs) (33)}$$

From (33) and $m_al_a\geq p(p-a)=r_br_c$(and analogs)(Panaitopol) we obtain:

$$\frac{n_a}{h_a}\geq\frac{\sqrt{r^2+l_a(p_a-m_a)}}{r}\text{(and analogs) (34)}$$

From (33) and $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$ after summation we obtain:

$$1\geq\sum\frac{\sqrt{r^2+p_al_a-r_br_c}}{n_a}\text{ (35)}$$

From (35) and $m_al_a\geq p(p-a)=r_br_c$(and analogs)(Panaitopol) we obtain:

$$1\geq\sum\frac{\sqrt{r^2+l_a(p_a-m_a)}}{n_a}\text{(36)}$$

From (33) $\rightarrow\frac{n_a}{\sqrt{r^2+p_al_a-r_br_c}}\geq\frac{h_a}{r}$ and after summation we obtain:

$$\sum\frac{n_a}{\sqrt{r^2+p_al_a-r_br_c}}\geq\frac{h_a+h_b+h_c}{r}\text{ (37)}$$

$2S=2pr=ah_a\rightarrow(a+b+c)r=ah_a\rightarrow\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs)

$$\rightarrow\frac{n_a}{\sqrt{r^2+p_al_a-r_br_c}}\geq1+\frac{b+c}{a}\text{ (and analogs)(38)}$$

From (38) we obtain:

$$\prod\left(\frac{n_a}{\sqrt{r^2+p_al_a-r_br_c}}-1\right)\geq\frac{(a+b)(b+c)(a+c)}{abc}\text{(39)}$$

From $l_a=\frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)}=\frac{2\sqrt{bc}}{b+c}\sqrt{r_br_c}\rightarrow l_al_bl_c=\frac{8abc}{(a+b)(b+c)(a+c)}r_ar_br_c$ and using (39)

we obtain:

$$\frac{1}{8}\prod\left(\frac{n_a}{\sqrt{r^2+p_al_a-r_br_c}}-1\right)\geq\frac{r_ar_br_c}{l_al_bl_c}\text{(40)}$$

$\frac{R}{2r}=\frac{r_a r_b r_c}{h_a h_b h_c}=\frac{r_a r_b r_c}{l_a l_b l_c}\frac{l_a l_b l_c}{h_a h_b h_c}$ and using (40) we obtain:

$$\mathbf{\frac{l_a l_b l_c}{h_a h_b h_c}\prod\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq \frac{4R}{r}\ (41)}$$

We proved $\frac{R}{r}\geq\frac{m_b}{h_c}+\frac{m_c}{h_b}$ (and analogs)[8] and using (41) we obtain:

$$\mathbf{\frac{l_a l_b l_c}{h_a h_b h_c}\prod\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq 4\max\left\{\frac{m_b}{h_c}+\frac{m_c}{h_b},\frac{m_b}{h_a}+\frac{m_a}{h_b},\frac{m_a}{h_c}+\frac{m_c}{h_a}\right\}(42)}$$

From $r_a=\frac{S}{p-a}$; $p-a=\frac{b+c-a}{2}\to r_a=\frac{2S}{b+c-a}$;$2S=ah_a\to\frac{r_a}{h_a}=\frac{a}{b+c-a}$ (and analogs)

$\frac{h_a}{r_a}=\frac{b+c-a}{a}\to\frac{b+c}{a}=1+\frac{h_a}{r_a}$ (and analogs) and using (38) we obtain:

$$\mathbf{\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\geq\frac{h_a}{r_a}\ (\text{and analogs})(43)}$$

$$\mathbf{\frac{r_a}{h_a}\geq\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}(\text{and analogs})\to\frac{R}{2r}\geq\prod\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}(44)}$$

$\sin^2\frac{A}{2}=\frac{r}{2R}\frac{r_a}{h_a}=\frac{r_a-r}{4R}$ (and analogs) $\to\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogs)

$r_a+r_b+r_c=4R+r\to\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$,we obtain:

$$\mathbf{\frac{2R}{r}\geq1+\sum\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}(45)}$$

From (44) and (45) we obtain:

$\left(\frac{R}{r}\right)^2\geq q_1q_2$ (46) where $q_1=\prod\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}$

$$q_2=1+\sum\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}$$

From $\frac{r_a}{h_a}\geq\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}$ (and analogs) and $r_a+r_b+r_c=4R+r$ we obtain: $4R+$

$$\mathbf{r\geq\sum h_a\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}(47)}$$

Can be proved that: $\frac{b+c}{a}=\frac{r_a+r}{r_a-r}$ (and analogs) and using (38) we obtain:

$$\mathbf{\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)(r_a-r)\geq r_a+r\ (\text{and analogs})\ (48)}$$

From (48) after summation we obtain:

$$\sum\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)(r_a-r)\geq 4(R+r)\ (49)$$

Now we use the well-known relation:

$\cos\frac{B-C}{2}=\frac{h_a}{l_a}=\frac{b+c}{a}\sin\frac{A}{2}$(and analogs) and using (38) we obtain:

$$\sin\frac{A}{2}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\frac{h_a}{l_a}\ \text{(and analogs) (50)}$$

From (50) after summation we obtain:

$$\sum\sin\frac{A}{2}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\sum\frac{h_a}{l_a}(51)$$

We use $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs) [10] and the inequality:

$\sqrt{\frac{2R}{r}}\geq\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}$ [11], we obtain:

$$\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\sqrt{\frac{h_a}{r_a}}\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}\right)\text{(and analogs) (52)}$$

From (52) after summation we obtain:

$$\sum\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}\right)\sum\sqrt{\frac{h_a}{r_a}}(53)$$

Using (38), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs),

$\sqrt{\frac{2R}{r}}\geq\frac{p\sqrt{3}}{h_a+h_b+h_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c}$ [11] we obtain:

$$\sum\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\left(\frac{p\sqrt{3}}{h_a+h_b+h_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c}\right)\sum\sqrt{\frac{h_a}{r_a}}(54)$$

Using (38), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs),

$\sqrt{\frac{2R}{r}}\geq\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c}$ [11], we obtain:

$$\sum\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c},\right)\sum\sqrt{\frac{h_a}{r_a}}(55)$$

Using (38), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}\frac{h_a}{l_a}}\sqrt{\frac{h_a}{r_a}}$ (and analogs), $\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}$(and analogs) [11] we obtain:

$$\sum \frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a - r_b r_c}} - 1\right) \geq \sum \sqrt{\frac{h_a}{r_a}} \left(\sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}\right) (56)$$

From (15), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}\frac{h_a}{l_a}}\sqrt{\frac{h_a}{r_a}}$ (and analogs) we obtain:

$$\sqrt{\frac{R}{2r}}\ |b\text{-}c\ | \geq \frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}(p_a l_a - h_a^2)} \text{(and analogs) (57)}$$

From (57) after summation we obtain:

$$\sqrt{\frac{R}{2r}}(\ |b-c\ |+\ |a-c\ |+\ |b-a\ |) \geq \sum \frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}(p_a l_a - h_a^2)} (58)$$

$\frac{1}{2}(\ |b-c\ |+\ |a-c\ |+\ |b-a\ |) = \max(a, b, c) - \min(a, b, c)$ we obtain:

$$\sqrt{\frac{2R}{r}}[\max(a, b, c)\ - \min(a, b, c)\ ] \geq \sum \frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}(p_a l_a - h_a^2)} (59)$$

From (9) and $2\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}} - 1\right) \geq \frac{\sin(A+\omega)}{\sin\omega}$(and analogs)[7] we obtain:

$$2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}} \geq 1 + \frac{4p_a l_a + a^2}{4bc} \text{(and analogs) (60)}$$

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