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U.2528 If $a, b, c, x, y, m \in R_+^*$, then in any triangle ABC holds the inequality:

$$\frac{a^{m+2}}{(xa+yb)^m}+\frac{b^{m+2}}{(xb+yc)^m}+\frac{c^{m+2}}{(xc+ya)^m}\geq\frac{2(s^2-r^2-4Rr)}{(x+y)^m}$$

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Solution by Titu Zvonaru-Romania

Applying Radon's inequality, the known inequality $ab+bc+ca\leq a^2+b^2+c^2$

and the formula $a^2+b^2+c^2=2(s^2-r^2-4Rr)$, it follows that:

$$\frac{a^{m+2}}{(xa+yb)^m}+\frac{b^{m+2}}{(xb+yc)^m}+\frac{c^{m+2}}{(xc+ya)^m}=$$

$$=\frac{a^{2m+2}}{(xa^2+yab)^m}+\frac{b^{2m+2}}{(xb^2+ybc)^m}+\frac{c^{2m+2}}{(xc^2+yca)^m}=$$

$$=\frac{(a^2)^{m+1}}{(xa^2+yab)^m}+\frac{(b^2)^{m+1}}{(xb^2+ybc)^m}+\frac{(c^2)^{m+1}}{(xc^2+yca)^m}\geq$$

$$\geq\frac{(a^2+b^2+c^2)^{m+1}}{(x(a^2+b^2+c^2)+y(ab+bc+ca))^m}\geq$$

$$=\frac{(a^2+b^2+c^2)^{m+1}}{\left(x(a^2+b^2+c^2)+y(a^2+b^2+c^2)\right)^m}=\frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(a^2+b^2+c^2)^m}=$$

$$=\frac{a^2+b^2+c^2}{(x+y)^m}=\frac{2(s^2-r^2-4Rr)}{(x+y)^m}$$

Equality holds if and only if the triangle ABC is equilateral.