ROMANIAN MATHEMATICAL MAGAZINE

U.2529 If $x, y \in R_+^*$, then in any triangle ABC holds the inequality:

$$\frac{a^3}{xa+yb}+\frac{b^3}{xb+yc}+\frac{c^3}{xc+ya}\geq\frac{2(s^2-r^2-4Rr)}{x+y}$$

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Solution by Titu Zvonaru-Romania

Applying Radon's inequality, the known inequality $ab+bc+ca\leq a^2+b^2+c^2$ and the formula $a^2+b^2+c^2=2(s^2-r^2-4Rr)$, it follows that

$$\frac{a^3}{xa+yb}+\frac{b^3}{xb+yc}+\frac{c^3}{xc+ya}=\frac{a^4}{xa^2+yab}+\frac{b^4}{xb^2+ybc}+\frac{c^4}{xc^2+yca}=$$

$$=\frac{(a^2)^2}{xa^2+yab}+\frac{(b^2)^2}{xb^2+ybc}+\frac{(c^2)^2}{xc^2+yca}\geq\frac{(a^2+b^2+c^2)^2}{x(a^2+b^2+c^2)+y(ab+bc+ca)}\geq$$

$$=\frac{(a^2+b^2+c^2)^2}{(x+y)(a^2+b^2+c^2)}=\frac{a^2+b^2+c^2}{x+y}=\frac{2(s^2-r^2-4Rr)}{x+y}.$$

Equality holds if and only if the triangle ABC is equilateral.