U.2530 If $a,b,c\in R_+^*$, $m\in R_+$, then in any triangle ABC holds the inequality:

$$\frac{r_a^{m+2}}{(xr_a+yr_b)^m}+\frac{r_b^{m+2}}{(xr_b+yr_c)^m}+\frac{r_c^{m+2}}{(xr_c+yr_a)^m}\ge\frac{(4R+r)^2-2s^2}{(x+y)^m}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Titu Zvonaru-Romania

Applying Radon' inequality and the known inequality

$ab+bc+ca\le a^2+b^2+c^2$, it follows that

$$\frac{r_a^{m+2}}{(xr_a+yr_b)^m}+\frac{r_b^{m+2}}{(xr_b+yr_c)^m}+\frac{r_c^{m+2}}{(xr_c+yr_a)^m}=$$

$$=\frac{r_a^{2m+2}}{(xr_a^2+yr_ar_b)^m}+\frac{r_b^{2m+2}}{(xr_b^2+yr_br_c)^m}+\frac{r_c^{2m+2}}{(xr_c^2+yr_cr_a)^m}=$$

$$=\frac{(r_a^2)^{m+1}}{(xr_a^2+yr_ar_b)^m}+\frac{(r_b^2)^{m+1}}{(xr_b^2+yr_br_c)^m}+\frac{(r_c^2)^{m+1}}{(xr_c^2+yr_cr_a)^m}$$

$$\ge\frac{(r_a^2+r_b^2+r_c^2)^{m+1}}{\left(x(r_a^2+r_b^2+r_c^2)+y(r_ar_b+r_br_c+r_cr_a)\right)^m}\ge$$

$$\ge\frac{(r_a^2+r_b^2+r_c^2)^{m+1}}{\left(x(r_a^2+r_b^2+r_c^2)+y(r_a^2+r_b^2+r_c^2)\right)^m}=$$

$$=\frac{(r_a^2+r_b^2+r_c^2)^{m+1}}{(x+y)^m(r_a^2+r_b^2+r_c^2)^m}=\frac{r_a^2+r_b^2+r_c^2}{(x+y)^m}\quad(1)$$

Since

$$r_a+r_b+r_c=F\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)=$$

$$=\frac{F\left((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\right)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs(3s^2-s(a+b+b+c+c+a)+ab+bc+ca)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs(3s^2-4s^2+s^2+r^2+4Rr)}{F^2}=\frac{sr(4R+r)}{sr}=4R+r$$

and

$$r_ar_b+r_br_c+r_cr_a=$$

$$F^2\left(\frac{1}{(s-a)(s-b)}+\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}\right)=$$

$$=\frac{F^2s(s-c+s-a+s-b)}{s(s-a)(s-b)(s-c)}=\frac{F^2s^2}{F^2}=s^2,$$

we obtain

$$r_a^2+r_b^2+r_c^2=(4R+r)^2-2s^2 \quad (2)$$

By (1) and (2) it results that

$$\frac{r_a^{m+2}}{(xr_a+yr_b)^m}+\frac{r_b^{m+2}}{(xr_b+yr_c)^m}+\frac{r_c^{m+2}}{(xr_c+yr_a)^m}\geq\frac{(4R+r)^2-2s^2}{(x+y)^m}.$$

Equality holds if and only if the triangle ABC is equilateral.