ROMANIAN MATHEMATICAL MAGAZINE

U.2531 If $m \geq 0$ and $x, y, z > 0$, then

$$\frac{x^{4m+3}}{x+2y}+\frac{y^{4m+3}}{y+2z}+\frac{z^{4m+3}}{z+2x} \geq (x^2y^2z^2)^{\frac{2m+1}{3}}$$

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Applying Radon's inequality and $AM-GM$ inequality, it follows that

$$\frac{x^{4m+3}}{x+2y}+\frac{y^{4m+3}}{y+2z}+\frac{z^{4m+3}}{z+2x}=\frac{x^{4m+4}}{x^2+2xy}+\frac{y^{4m+4}}{y^2+2yz}+\frac{z^{4m+4}}{z^2+2zx}=$$

$$=\frac{(x^{2m+2})^2}{x^2+2xy}+\frac{(y^{2m+2})^2}{y^2+2yz}+\frac{(z^{2m+2})^2}{z^2+2zx} \geq \frac{(x^{2m+2}+y^{2m+2}+y^{2m+2})^2}{x^2+y^2+z^2+2xy+2yz+2zx} \quad (1)$$

By Power Mean inequality we obtain:

$$\left(\frac{x^{2m+2}+y^{2m+2}+z^{2m+2}}{3}\right)^{\frac{1}{2(m+1)}} \geq \frac{x+y+z}{3}$$

$$\Leftrightarrow 3^{2m+1}(x^{2m+2}+y^{2m+2}+z^{2m+2}) \geq (x+y+z)^{2(m+1)} \quad (2)$$

Using (1) and (2) it results that

$$\frac{x^{4m+3}}{x+2y}+\frac{y^{4m+3}}{y+2z}+\frac{z^{4m+3}}{z+2x} \geq \frac{(x^{2m+2}+y^{2m+2}+y^{2m+2})^2}{(x+y+z)^2} \geq$$

$$\geq \frac{(x+y+z)^{4m+4}}{3^{4m+2}(x+y+z)^2}=\frac{(x+y+z)^{4m+2}}{3^{4m+2}} \geq \frac{3^{4m+2}(xyz)^{\frac{4m+2}{3}}}{3^{4m+2}}=(x^2y^2z^2)^{\frac{2m++}{3}}.$$

Equality holds if and only if $x=y=z$.