ROMANIAN MATHEMATICAL MAGAZINE

U.2532 If $x,y\in R_+^*$, then in any triangle ABC holds the inequality:

$$\frac{r_a^3}{xr_a+yr_b}+\frac{r_b^3}{xr_b+yr_c}+\frac{r_c^3}{xr_c+yr_a}\ge\frac{(4R+r)^2-2s^2}{x+y}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Titu Zvonaru-Romania

Applying Radon's inequality and the known inequality $ab+bc+ca\le a^2+b^2+c^2$, it follows that

$$\frac{r_a^3}{xr_a+yr_b}+\frac{r_b^3}{xr_b+yr_c}+\frac{r_c^3}{xr_c+yr_a}=\frac{r_a^4}{xr_a^2++yr_ar_b}+\frac{r_b^4}{xr_b^2+yr_br_c}+\frac{r_c^4}{xr_c^2+yr_cr_a}=$$

$$=\frac{(r_a^2)^2}{xr_a^2+yr_ar_b}+\frac{(r_b^2)^2}{xr_b^2+yr_br_c}+\frac{(r_c^2)^2}{xr_c^2+yr_cr_a}\ge$$

$$\ge\frac{(r_a^2+r_b^2+r_c^2)^2}{x(r_a^2+r_b^2+r_c^2)+y(r_ar_b+r_br_c+r_cr_a)}\ge\frac{(r_a^2+r_b^2+r_c^2)^2}{(x+y)(r_a^2+r_b^2+r_c^2)}=\frac{r_a^2+r_b^2+r_c^2}{x+y}\quad(1)$$

Since

$$r_a+r_b+r_c=F\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)=$$

$$=\frac{F\big((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\big)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs\big(3s^2-s(a+b+b+c+c+a)+ab+bc+ca\big)}{s(s-a)(s-b)(s-c)}=\frac{Fs\big(3s^2-4s^2+s^2+r^2+4Rr\big)}{F^2}=$$

$$=\frac{sr(4R+r)}{sr}=4R+r$$

$$r_ar_b+r_br_c+r_cr_a=F^2\left(\frac{1}{(s-a)(s-b)}+\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}\right)=$$

$$=\frac{F^2s(s-c+s-a+s-b)}{s(s-a)(s-b)(s-c)}=\frac{F^2s^2}{F^2}=s^2,$$

we obtain

$$r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - 2s^2 \quad (2)$$

By (1) and (2) it results that

$$\frac{r_a^3}{xr_a + yr_b} + \frac{r_b^3}{xr_b + yr_c} + \frac{r_c^3}{xr_c + yr_a} \geq \frac{(4R + r)^2 - 2s^2}{x + y}.$$

Equality holds if and only if the triangle ABC is equilateral.