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U.2533 If $x, y \in R_+^*$, then in any triangle ABC holds the inequality:

$$\frac{\mathrm{tg}^2\frac{A}{2}}{x\mathrm{tg}\frac{A}{2}+y\mathrm{tg}\frac{B}{2}}+\frac{\mathrm{tg}^2\frac{B}{2}}{x\mathrm{tg}\frac{B}{2}+y\mathrm{tg}\frac{C}{2}}+\frac{\mathrm{tg}^2\frac{C}{2}}{x\mathrm{tg}\frac{C}{2}+y\mathrm{tg}\frac{A}{2}}\geq\frac{4R+r}{(x+y)s}$$

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Solution by Titu Zvonaru-Romania

Since $\mathrm{tg}\frac{A}{2}=\frac{r}{s-a}$, we obtain

$$\mathrm{tg}\frac{A}{2}+\mathrm{tg}\frac{B}{2}+\mathrm{tg}\frac{C}{2}=r\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)=$$

$$=\frac{r\big((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\big)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{rs(3s^2-s(a+b+b+c+c+a)+ab+bc+ca)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{F(3s^2-4s^2+s^2+r^2+4Rr)}{F^2}=\frac{r(4R+r)}{sr}=\frac{4R+r}{s}\quad(1)$$

Applying Bergström's inequality and (1), it follows that

$$\frac{\mathrm{tg}^2\frac{A}{2}}{x\mathrm{tg}\frac{A}{2}+y\mathrm{tg}\frac{B}{2}}+\frac{\mathrm{tg}^2\frac{B}{2}}{x\mathrm{tg}\frac{B}{2}+y\mathrm{tg}\frac{C}{2}}+\frac{\mathrm{tg}^2\frac{C}{2}}{x\mathrm{tg}\frac{C}{2}+y\mathrm{tg}\frac{A}{2}}\geq$$

$$\geq\frac{\left(\mathrm{tg}\frac{A}{2}+\mathrm{tg}\frac{B}{2}+\mathrm{tg}\frac{C}{2}\right)^2}{(x+y)\left(\mathrm{tg}\frac{A}{2}+\mathrm{tg}\frac{B}{2}+\mathrm{tg}\frac{C}{2}\right)}=\frac{4R+r}{(x+y)s}$$

Equality holds if and only if the triangle ABC is equilateral.