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U.2534 In any triangle ABC holds the inequality:

$$\frac{\operatorname{ctg}^2\frac{A}{2}}{2s-\left(\operatorname{ctg}\frac{A}{2}-\operatorname{ctg}\frac{B}{2}\right)}+\frac{\operatorname{ctg}^2\frac{B}{2}}{2s-\left(\operatorname{ctg}\frac{B}{2}-\operatorname{ctg}\frac{C}{2}\right)}+\frac{\operatorname{ctg}^2\frac{C}{2}}{2s-\left(\operatorname{ctg}\frac{C}{2}-\operatorname{ctg}\frac{A}{2}\right)}\geq\frac{s}{6r^2}$$

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Solution by Titu Zvonaru-Romania

Since $\operatorname{ctg}\frac{A}{2}=\frac{s-a}{r}$ we obtain

$$\operatorname{ctg}\frac{A}{2}+\operatorname{ctg}\frac{B}{2}+\operatorname{ctg}\frac{C}{2}=\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}=\frac{s}{r}\quad(1)$$

Applying Bergström's inequality and (1), it follows that:

$$\frac{\operatorname{ctg}^2\frac{A}{2}}{2s-\left(\operatorname{ctg}\frac{A}{2}-\operatorname{ctg}\frac{B}{2}\right)}+\frac{\operatorname{ctg}^2\frac{B}{2}}{2s-\left(\operatorname{ctg}\frac{B}{2}-\operatorname{ctg}\frac{C}{2}\right)}+\frac{\operatorname{ctg}^2\frac{C}{2}}{2s-\left(\operatorname{ctg}\frac{C}{2}-\operatorname{ctg}\frac{A}{2}\right)}\geq$$

$$\geq\frac{\left(\operatorname{ctg}\frac{A}{2}+\operatorname{ctg}\frac{B}{2}+\operatorname{ctg}\frac{C}{2}\right)^2}{6s-\operatorname{ctg}\frac{A}{2}+\operatorname{ctg}\frac{B}{2}-\operatorname{ctg}\frac{B}{2}+\operatorname{ctg}\frac{C}{2}-\operatorname{ctg}\frac{C}{2}+\operatorname{ctg}\frac{A}{2}}=\frac{\left(\frac{s}{r}\right)^2}{6s}=\frac{s}{6r^2}.$$

Equality holds if and only if the triangle ABC is equilateral.