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U.2539 In any triangle ABC, g_a Gergonne's cevian, holds:

$$\left(g_a^2+2\right)\left(g_b^2+2\right)\left(g_c^2+2\right) \geq 36\left(\frac{F}{R}\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

Solution by Titu Zvonaru-Romania

We have $g_a \geq h_a$ and $ah_a = 2F$. Applying Arkady Alt's inequality

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

(with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$) and the inequality:

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{\sqrt{3}}{R}$ (item 5.23 from $[1]$), it follows that:

$$\left(g_a^2+2\right)\left(g_b^2+2\right)\left(g_c^2+2\right) \geq \left(h_a^2+2\right)\left(h_b^2+2\right)\left(h_c^2+2\right) \geq$$

$$\geq \frac{3}{4}\left(\sqrt{2}\right)^2(h_a+h_b+h_c)^2 = 3\left(\frac{2F}{a}+\frac{2F}{b}+\frac{2F}{c}\right)^2 =$$

$$= 12F^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \geq 12F^2\left(\frac{\sqrt{3}}{R}\right)^2 = 36\left(\frac{F}{R}\right)^2.$$

Equality holds if and only if the triangle ABC is equilateral and $h_a = 1$,

that is $a = b = c = \frac{2\sqrt{3}}{3}$.

$[1]$ O. Bottema, Geometric Inequalities, Groningen 1969

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2\left((x+y)^2+t^2\right) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.