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U.2540 In any triangle ABC, n_a - Nagel's cevian, holds:

$$(n_a^4+2)(n_b^4+2)(n_c^4+2)\ge 48\left(\frac{F}{R}\right)^4$$

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We have $n_a\ge h_a$ and $ah_a=2F$. Applying Arkady Alt's inequality:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

(with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$), the known inequality:

$$3(x^2+y^2+z^2)\ge(a+b+c)^2$$

and the inequality $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{\sqrt{3}}{R}$ (item 5.23 from $[1]$), it follows that:

$$(n_a^4+2)(n_b^4+2)(n_c^4+2)\ge(h_a^4+2)(h_b^4+2)(h_c^4+2)\ge\frac{3}{4}\left(\sqrt{2}\right)^2\left(h_a^2+h_b^2+h_c^2\right)^2=$$

$$=3\left(\frac{4F^2}{a^2}+\frac{4F^2}{b^2}+\frac{4F}{c^2}\right)^2=48F^4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2\ge 48F^4\cdot\frac{1}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^4\ge$$

$$\ge 48F^4\cdot\frac{1}{9}\left(\frac{\sqrt{3}}{R}\right)^4=48\left(\frac{F}{R}\right)^4.$$

Equality holds if and only if the triangle ABC is equilateral

and $h_a^2=1$ that is $a=b=c=\sqrt{\frac{2\sqrt{3}}{3}}$.

$[1]$ O. Bottema, Geometric Inequalities, Groningen 1969

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\ge\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}\left((x+y)^2+t^2\right)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}\left(t(x+y)+tz\right)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.