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U.2545 In any triangle ABC, g_a – Gergonne's cevians, n_a - Nagel's cevians, holds:

$$\frac{g_a n_b}{h_c^4}+\frac{g_b n_c}{h_a^4}+\frac{g_c n_a}{h_b^4}\geq\frac{\sqrt{3}}{F}$$

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Solution by Titu Zvonaru-Romania

We have $g_a, n_a \geq h_a$ and $ah_a = 2F$. Applying Bergström's inequality, the known inequality $ab+bc+ca\leq a^2+b^2+c^2$ and Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\geq 4\sqrt{3}F$, it follows that

$$\frac{g_a n_b}{h_c^4}+\frac{g_b n_c}{h_a^4}+\frac{g_c n_a}{h_b^4}\geq\frac{h_a h_b}{h_c^4}+\frac{h_b h_c}{h_a^4}+\frac{h_c h_a}{h_b^4}=$$

$$=\frac{ah_a bh_b c^4}{abc^4 h_c^4}+\frac{bh_b ch_c a^4}{bca^4 h_a^4}+\frac{ch_c ah_a b^4}{cab^4 h_b^4}=\frac{4F^2}{16F^4}\left(\frac{c^4}{ab}+\frac{a^4}{bc}+\frac{b^4}{ca}\right)\geq$$

$$\geq\frac{1}{4F^2}\cdot\frac{(a^2+b^2+c^2)^2}{ab+bc+ca}\geq\frac{1}{4F^2}(a^2+b^2+c^2)\geq\frac{4\sqrt{3}F}{4F^2}=\frac{\sqrt{3}}{F}.$$

Equality holds if and only if the triangle ABC is equilateral.