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U.2549 If $m \geq 0$, then in triangle ABC holds:

$$\frac{h_a^m}{(h_a-2r)^m}+\frac{h_b^m}{(h_b-2r)^m}+\frac{h_c^m}{(h_c-2r)^m} \geq 3^{m+1}$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

Solution by Titu Zvonaru-Romania

We have $ah_a = bh_b = ch_c = 2F$. Applying Radon's inequality it follows that

$$\frac{h_a^m}{(h_a-2r)^m}+\frac{h_b^m}{(h_b-2r)^m}+\frac{h_c^m}{(h_c-2r)^m}=$$

$$=\frac{a^m h_a^m}{(ah_a-2ar)^m}+\frac{b^m h_b^m}{(bh_b-2br)^m}+\frac{c^m h_c^m}{(ch_c-2cr)^m}=$$

$$=(2F)^m\left(\frac{1^{m+1}}{(2F-2ar)^m}+\frac{1^{m+1}}{(2F-2br)^m}+\frac{1^{m+1}}{(2f-2cr)^m}\right)\geq$$

$$\geq (2F)^m\cdot\frac{(1+1+1)^{m+1}}{\big(6F-2r(a+b+c)\big)^m}=(2F)^m\cdot\frac{3^{m+1}}{(6F-4F)^m}=3^{m+1}$$

Equality holds if and only if the triangle ABC is equilateral.