ROMANIAN MATHEMATICAL MAGAZINE

U.2552 If $x, y, z > 0, t \geq 0$, then in any triangle ABC holds

$$\frac{x+t}{y+z+2t}\cdot a+\frac{y+t}{z+x+2t}\cdot b+\frac{z+t}{x+y+2t}\cdot c\geq (27)^{1/4}\sqrt{F}$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

Solution by Titu Zvonaru-Romania

Denoting $m=x+t, n=y+t, q=z+t$, the given inequality becomes

$$\frac{m}{n+q}\cdot a+\frac{n}{q+m}\cdot b+\frac{q}{m+n}\cdot c\geq (27)^{\frac{1}{4}}\sqrt{F}\quad (1)$$

We consider the triangle $A_1B_1C_1$ with the sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$ and the area F_1.

By a theorem of Mehmet Şahin (Turkey), which appeared in AMM (2015),

there is true the formula:

$$F_1=\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}\quad (2)$$

Applying Tsintsifas' inequality for the triangle with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$, it follows that

$$\frac{m}{n+q}\cdot a+\frac{n}{q+m}\cdot b+\frac{q}{m+n}\cdot c\geq 2\sqrt{3}F_1\quad (3)$$

Using $(1), (2), (3)$ and the known inequality $r_a+r_b+r_c\geq\sqrt{3}s$ it results that

$$\frac{x+t}{y+z+2t}\cdot a+\frac{y+t}{z+x+2t}\cdot b+\frac{z+t}{x+y+2t}\cdot c\geq 2\sqrt{3}F_1=$$

$$=2\sqrt{3}\cdot\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}\geq\sqrt{3}\sqrt{\sqrt{3}rs}=(27)^{\frac{1}{4}}\sqrt{F}.$$

Equality holds if and only if $a=b=c, m=n=q$, that is $x=y=z$.