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U.2553 In any triangle ABC the following relationship holds:

$$\frac{(h_a-r)^2}{h_a(h_a+r)}+\frac{(h_b-r)^2}{h_b(h_b+r)}+\frac{(h_c-r)^2}{h_c(h_c+r)}\geq 1$$

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We have $ah_a=bh_b=ch_c=2F$. Applying Bergström's inequality, it follows that

$$\frac{(h_a-r)^2}{h_a(h_a+r)}+\frac{(h_b-r)^2}{h_b(h_b+r)}+\frac{(h_c-r)^2}{h_c(h_c+r)}=$$

$$=\frac{(ah_a-ar)^2}{ah_a(ah_a+ar)}+\frac{(bh_b-br)^2}{bh_b(bh_b+br)}+\frac{(ch_c-cr)^2}{ch_c(ch_c+cr)}=$$

$$=\frac{1}{2F}\left(\frac{(2F-ar)^2}{2F+ar}+\frac{(2F-br)^2}{2F+br}+\frac{(2F-cr)^2}{2F+cr}\right)^2\geq$$

$$\geq\frac{1}{2F}\cdot\frac{\big(6F-r(a+b+c)\big)^2}{6F+r(a+b+c)}=\frac{16F^2}{16F^2}=1.$$

Equality holds if and only if the triangle ABC is equilateral.