ROMANIAN MATHEMATICAL MAGAZINE

U.2558 If $x, y, z > 0$ then:

$$2(1+\sqrt{2})\sum_{cyc}\frac{xy}{\sqrt{x^2+y^2}} \leq 2(x+y+z)+\sum_{cyc}\sqrt{x^2+y^2}$$

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By $QM-AM$ inequality we have $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$; then $\sqrt{x^2+y^2} \geq \frac{x+y}{\sqrt{2}}$.

Applying $HM-AM$ inequality $\frac{2xy}{x+y} \leq \frac{x+y}{2}$, it follows that:

$$2(1+\sqrt{2})\sum_{cyc}\frac{xy}{\sqrt{x^2+y^2}} \leq 2(1+\sqrt{2})\sum_{cyc}\frac{xy\sqrt{2}}{x+y} = 2(\sqrt{2}+2)\sum_{cyc}\frac{xy}{x+y} =$$

$$= \frac{2(\sqrt{2}+2)}{2}\sum_{cyc}\frac{2xy}{x+y} \leq \frac{2(\sqrt{2}+2)}{2}\sum_{cyc}\frac{x+y}{2} =$$

$$= 2(x+y+z)+\sum_{cyc}\frac{x+y}{\sqrt{2}} \leq 2(x+y+z)+\sum_{cyc}\sqrt{x^2+y^2}.$$

Equality holds if and only if $x = y = z$.