ROMANIAN MATHEMATICAL MAGAZINE

U.2566 If $x, y, z >$ then:

$$(x+y)(2z+x+y)(y+z)(2x+y+z)(z+x)(2y+z+x) \geq 512x^2y^2z^2$$

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Solution 1: Applying Cesaro's inequality it follows that:

$$(x+y)(y+z)(z+x) \geq 8xyz \quad (1)$$

$$(2z+x+y)(2x+y+z)(2y+z+x) =$$

$$= \big((x+y)+(x+z)\big)\big((x+z)+(y+z)\big)\big((y+z)+(x+y)\big) \geq$$

$$\geq 8(x+y)(y+z)(z+x) \geq 64xyz \quad (2)$$

Multiplying (1) and (2) it results that

$$(x+y)(2z+x+y)(y+z)(2x+y+z)(z+x)(2y+z+x) \geq$$

$$\geq (8xyz)(64xyz) = 512x^2y^2z^2.$$

Equality holds if and only if $x = y = z$.

Solution 2. Applying $AM-GM$ inequality we get:

$$y+z \geq 2(yz)^{\frac{1}{2}}, 2x+y+z = x+x+y+z \geq 4(x^2yz)^{\frac{1}{4}},$$

Hence

$$(y+z)(2x+y+z) \geq 8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}} \quad (3)$$

and similarly

$$(z+x)(2y+z+x) \geq 8y^{\frac{1}{2}}z^{\frac{3}{4}}x^{\frac{3}{4}} \quad (4) \quad (x+y)(2z+x+y) \geq 8z^{\frac{1}{2}}x^{\frac{3}{4}}y^{\frac{3}{4}} \quad (5)$$

Since $\frac{1}{2}+\frac{3}{4}+\frac{3}{4} = 2$, mutiplying $(3), (4)$ and (5) yields

$$(x+y)(2z+x+y)(y+z)(2x+y+z)(z+x)(2y+z+x) \geq 512x^2y^2z^2.$$