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U.2567 If M in an interior point in the triangle $ABC, d_a = d(M, BC)$,

$d_b = d(M, CA), d_c = d(M, AB)$ then:

$$\frac{a^3}{d_a + h_a} + \frac{b^3}{d_b + h_b} + \frac{c^3}{d_c + h_c} \geq 6F$$

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Solution by Titu Zvonaru-Romania

We have $ad_a + bd_b + cd_c = 2F, ah_a = bh_b = ch_c = 2F$.

Applying Bergström's inequality and Ionescu-Weitzenbock's inequality

$a^2 + b^2 + c^2 \geq 4\sqrt{3}F$, it follows that:

$$\frac{a^3}{d_a + h_a} + \frac{b^3}{d_b + h_b} + \frac{c^3}{d_c + h_c} = \frac{a^4}{ad_a + ah_a} + \frac{b^4}{bd_b + bh_b} + \frac{c^4}{cd_c + ch_c} \geq$$

$$\geq \frac{(a^2 + b^2 + c^2)^2}{ad_a + bd_b + cd_c + ah_a + bh_b + ch_c} \geq \frac{\left(4\sqrt{3}F\right)^2}{8F} = 6F.$$

Equality holds if and only if the triangle ABC is equilateral.