ROMANIAN MATHEMATICAL MAGAZINE

U.2569 If $m,n,x,y,z>0$ then in any triangle ABC holds:

$$\left(\frac{mxa^4}{y+z}+\frac{nyb^4}{z+x}\right)^2+\left(\frac{myb^4}{z+x}+\frac{nzc^4}{x+y}\right)^2+\left(\frac{mzc^4}{x+y}+\frac{nxa^4}{y+z}\right)^2\geq\frac{64(m+n)^2}{3}F^4$$

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By the known inequality $3(x^2+y^2+z^2)\geq(x+y+z)^2$, we obtain:

$$\left(\frac{mxa^4}{y+z}+\frac{nyb^4}{z+x}\right)^2+\left(\frac{myb^4}{z+x}+\frac{nzc^4}{x+y}\right)^2+\left(\frac{mzc^4}{x+y}+\frac{nxa^4}{y+z}\right)^2\geq$$

$$\geq\frac{1}{3}\left(\frac{mxa^4}{y+z}+\frac{nyb^4}{z+x}+\frac{myb^4}{z+x}+\frac{nzc^4}{x+y}+\frac{mzc^4}{x+y}+\frac{nxa^4}{y+z}\right)^2=$$

$$=\frac{1}{3}\left(\frac{(m+n)xa^4}{y+z}+\frac{(m+n)yb^4}{z+x}+\frac{(n+n)zc^4}{x+y}\right)^2$$

$$=\frac{(m+n)^2}{3}\left(\frac{xa^4}{y+z}+\frac{yb^4}{z+x}+\frac{zc^4}{x+y}\right)^2\quad(1)$$

Applying Bergström's inequality and the formula

$16F^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$, it follows that

$$\frac{xa^4}{y+z}+\frac{yb^4}{z+x}+\frac{zc^4}{x+y}=\frac{xa^4}{y+z}+a^4+\frac{yb^4}{z+x}+b^4+\frac{zc^4}{x+y}+c^4-(a^4+b^4+c^4)=$$

$$=(x+y+z)\left(\frac{a^4}{y+z}+\frac{b^4}{z+x}+\frac{c^4}{x+y}\right)-(a^4+b^4+c^4)\geq$$

$$\geq\frac{(x+y+z)(a^2+b^2+c^2)^2}{2(x+y+z)}-(a^4+b^4+c^4)=\frac{(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)}{2}=$$

$$=\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{2}=8F^2\quad(2)$$

Using (1) and (2) it results that

$$\left(\frac{mxa^4}{y+z}+\frac{nyb^4}{z+x}\right)^2+\left(\frac{myb^4}{z+x}+\frac{nzc^4}{x+y}\right)^2+\left(\frac{mzc^4}{x+y}+\frac{nxa^4}{y+z}\right)^2\geq\frac{64(m+n)^2}{3}F^4$$

Equality holds if and only if the triangle ABC is equilateral and $x=y=z$.