ROMANIAN MATHEMATICAL MAGAZINE

U.2571. If $x > 0$ then in the triangle ABC holds:

$$\left(\frac{a^{2x+4}}{(bR+cr)^{2x}}+1\right)\left(\frac{b^{2x+4}}{(cR+ar)^{2x}}+1\right)\left(\frac{c^{2x+4}}{(aR+br)^{2x}}+1\right)\ge\frac{36}{(R+r)^{2x}}\cdot F^2$$

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Applying Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$

(with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$), it follows that

$$\left(\frac{a^{2x+4}}{(bR+cr)^{2x}}+1\right)\left(\frac{b^{2x+4}}{(cR+ar)^{2x}}+1\right)\left(\frac{c^{2x+4}}{(aR+br)^{2x}}+1\right)\ge$$

$$\ge\frac{3}{4}\left(\frac{a^{x+2}}{(bR+cr)^{x}}+\frac{b^{x+2}}{(cR+ar)^{x}}+\frac{c^{x+2}}{(aR+br)^{x}}\right)^2\quad(1)$$

Using Radon's inequality, the known inequality $ab+bc+ca\le a^2+b^2+c^2$

and Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\ge 4\sqrt{3}F$, we obtain

$$\frac{a^{x+2}}{(bR+cr)^{x}}+\frac{b^{x+2}}{(cR+ar)^{x}}+\frac{c^{x+2}}{(aR+br)^{x}}=\frac{(a^2)^{x+1}}{(abR+car)^{x}}+\frac{(b^2)^{x+1}}{(bcR+abr)^{x}}+\frac{(c^2)^{x+1}}{(caR+bcr)^{x}}\ge$$

$$\ge\frac{(a^2+b^2+c^2)^{x+1}}{\left(R(ab+bc+ca)+r(ab+bc+ca)\right)^x}\ge\frac{(a^2+b^2+c^2)^{x+1}}{(a^2+b^2+c^2)^x(R+r)^x}=$$

$$=\frac{a^2+b^2+c^2}{(R+r)^x}\ge\frac{4\sqrt{3}F}{(R+r)^x}\quad(2)$$

By (1) and (2) it results that

$$\left(\frac{a^{2x+4}}{(bR+cr)^{2x}}+1\right)\left(\frac{b^{2x+4}}{(cR+ar)^{2x}}+1\right)\left(\frac{c^{2x+4}}{(aR+br)^{2x}}+1\right)\ge\frac{3}{4}\left(\frac{4\sqrt{3}F}{(R+r)^x}\right)^2=\frac{36}{(R+r)^{2x}}\cdot F^2$$

Equality holds if and only if $a=b=c$ and $\frac{a^2}{R+r}=\frac{1}{\sqrt{2}}\Leftrightarrow\frac{a^2}{\frac{3a}{2\sqrt{3}}}=\frac{1}{\sqrt{2}}\Leftrightarrow a=\frac{\sqrt{3}}{2\sqrt{2}}$

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.