ROMANIAN MATHEMATICAL MAGAZINE

U.2575 If $x, y, z, t > 0, xyzt = 1$ then:

$$(x+y+z+t)^2 \le 2\left(xy+\frac{1}{xy}\right)\left(xz+\frac{1}{xz}\right)\left(xt+\frac{1}{xt}\right)$$

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Since $xyz = 1$, we have:

$$2\left(xy+\frac{1}{xy}\right)\left(xz+\frac{1}{xz}\right)\left(xt+\frac{1}{xt}\right)-(x+y+z+t)^2=$$

$$=2(xy+zt)(xz+yt)(xt+yz)-(x+y+z+t)^2=$$

$$=2(x^2yz+xy^2t+xz^2t+yzt^2)(xt+yz)-(x+y+z+t)^2=$$

$$=2x^2+2x^2y^2z^2+2x^2y^2t^2+2y^2+2x^2z^2t^2+2z^2+2t^2+2y^2z^2t^2-x^2-y^2-z^2-t^2-2xy-2xz-2xt-2yz-2yt-2zt=$$

$$=x^2y^2(z-t)^2+x^2t^2(y-z)^2+z^2t^2(x-y)^2+y^2z^2(t-x)^2+(x-z)^2+(y-t)^2 \ge 0.$$

Equality holds if and only if $x = y = z = t = 1$.