ROMANIAN MATHEMATICAL MAGAZINE

U.2576 If $t \geq 0, x, y > 0$, and in the triangle ABC, $m = m_a + m_b + m_c$ such that $mx > y\max\{m_a, m_b, m_c\}$, then holds:

$$\frac{m_a^{t+2}}{(xm - ym_a)^t} + \frac{m_b^{t+2}}{(xm - ym_b)^t} + \frac{m_c^{t+2}}{(xm - ym_c)^t} \geq \frac{3\sqrt{3}}{(3x - y)^t} \cdot F$$

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Applying Radon's inequality, the known inequality $xy + yz + zx \leq x^2 + y^2 + z^2$, the formula $m_a^2 + m_b^2 + m_c^2 = \frac{3(a^2+b^2+c^2)}{4}$ and Ionescu-Weitzenbock's inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$, it follows that:

$$\frac{m_a^{t+2}}{(xm - ym_a)^t} + \frac{m_b^{t+2}}{(xm - ym_b)^t} + \frac{m_c^{t+2}}{(xm - ym_c)^t} =$$

$$= \frac{m_a^{2t+2}}{(xmm_a - ym_a^2)^t} + \frac{m_b^{2t+2}}{\left(xmm_b - ym_b^2\right)^t} + \frac{m_c^{2t+2}}{(xmm_c - ym_c^2)^t} =$$

$$= \frac{(m_a^2)^{t+1}}{(xmm_a - ym_a^2)^t} + \frac{\left(m_b^2\right)^{t+1}}{\left(xmm_b - ym_b^2\right)^t} + \frac{(m_c^2)^{t+1}}{(xmm_c - ym_c^2)^t} \geq$$

$$\geq \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{t+1}}{\left(xm(m_a + m_b + m_c) - y\left(m_a^2 + m_b^2 + m_c^2\right)\right)^t} =$$

$$= \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{t+1}}{\left((x - y)\left(m_a^2 + m_b^2 + m_c^2\right) + 2x(m_a m_b + m_b m_c + m_c m_a)\right)^t} =$$

$$\geq \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{t+1}}{\left((x - y)\left(m_a^2 + m_b^2 + m_c^2\right) + 2x(m_a^2 + m_b^2 + m_c^2)\right)^t} = \frac{m_a^2 + m_b^2 + m_c^2}{(3x - y)^t} =$$

$$= \frac{3(a^2 + b^2 + c^2)}{4(3x - y)^t} \geq \frac{12\sqrt{3}F}{4(3x - y)^t} = \frac{3\sqrt{3}}{(3x - y)^t} \cdot F$$

Equality holds if and only if the triangle ABC is equilateral.