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U.2597 In any triangle ABC the following relationship holds:

$$(a^4+1)(b^4+1)(c^4+1)\geq\left(6F+\frac{\sqrt{3}}{4}((a-b)^2+(b-c)^2+(c-a)^2)\right)^2$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), it follows that:

$$(a^4+1)(b^4+1)(c^4+1)\geq\frac{3}{4}(a^2+b^2+c^2)^2.$$

It suffices to prove that

$$\frac{\sqrt{3}}{2}(a^2+b^2+c^2\geq 6F+\frac{\sqrt{3}}{4}((a-b)^2+(b-c)^2+(c-a)^2)$$

$$\frac{\sqrt{3}}{2}(a^2+b^2+c^2\geq 6F+\frac{\sqrt{3}}{4}(2a^2+2b^2+2c^2-2ab-2bc-2ca)$$

$$\sqrt{3}(ab+bc+ca)\geq 12F\Leftrightarrow ab+bc+ca\geq 4\sqrt{3}F \quad (1)$$

The inequality (1) is Gordon inequality (item 4.5 from $[1]$).

Equality holds if and only if the triangle ABC is equilateral and $a^2=\frac{1}{\sqrt{2}}$,

that is $a=b=c=(2)^{-1/4}$.

$[1]$ O. Bottema, *Geometric Inequalities*, Groningen 1969

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.