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U.2598 If $x,y,z>0$ then:

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\left(\frac{1}{(x+y)^4}+1\right)\left(\frac{1}{(y+z)^4}+1\right)\left(\frac{1}{(z+x)^4}+1\right)\ge\frac{729}{64}$$

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Applying twice Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$

(with egality if and only if $a=b=c=\frac{t}{\sqrt{2}}$) it follows that

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\ge\frac{3}{4}\left(\sqrt{2}\right)^2(xy+yz+zx)^2\quad(1)$$

and

$$\left(\frac{1}{(x+y)^4}+1\right)\left(\frac{1}{(y+z)^4}+1\right)\left(\frac{1}{(z+x)^4}+1\right)\ge$$

$$\ge\frac{3}{4}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}++\frac{1}{(z+x)^2}\right)^2\quad(2)$$

By (1) and (2) it results that

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\left(\frac{1}{(x+y)^4}+1\right)\left(\frac{1}{(y+z)^4}+1\right)\left(\frac{1}{(z+x)^4}+1\right)$$

$$\ge 3(xy+yz+zx)^2\cdot\frac{3}{4}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2=$$

$$=\frac{9}{4}(xy+yz+zx)^2\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2.$$

It remains to prove that

$$\frac{9}{4}(xy+yz+zx)^2\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2\ge\frac{729}{64}$$

$$(xy+yz+zx)^2\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2\ge\frac{81}{16}$$

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{9}{4}\quad(3)$$

The inequality (3) is a famous one – known as Iran 1996. In fact, it was proposed by Ji Chen in *Crux Mathematicorum* in april 1994.

Equality holds if and only if $x = t = z = 1$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3}{4} t^4 (x + y + z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2 + t^2)(y^2 + t^2) \geq \frac{3}{4} t^2 ((x + y)^2 + t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x - y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3t^2}{4}((x + y)^2 + t^2)(t^2 + z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x + y) + tz)^2 = \frac{3}{4} t^4 (x + y + z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.