ROMANIAN MATHEMATICAL MAGAZINE

U.2599 If $x, y, z > 0$ then:

$$\left(\frac{1}{(x+y)^4}+\frac{1}{(y+z)^4}+\frac{1}{(z+x)^4}\right)(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\geq\frac{81}{16}$$

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Applying Arkady Alt inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2$

(with egality if and only if $a=b=c=\frac{t}{\sqrt{2}}$) it follows that:

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\geq\frac{3}{4}\left(\sqrt{2}\right)^2(xy+yz+zx)^2 \quad (1)$$

By the known inequality $3(a^2+b^2+c^2)\geq(a+b+c)^2$ we obtain

$$\frac{1}{(x+y)^4}+\frac{1}{(y+z)^4}+\frac{1}{(z+x)^4}\geq\frac{1}{3}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2 \quad (2)$$

Multiplying (1) and (2) it results that

$$\left(\frac{1}{(x+y)^4}+\frac{1}{(y+z)^4}+\frac{1}{(z+x)^4}\right)(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\geq$$

$$\geq 3(xy+yz+zx)^2\cdot\frac{1}{3}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2.$$

It remains to prove that

$$(xy+yz+zx)^2\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2\geq\frac{81}{16}$$

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4} \quad (3)$$

The inequality (3) is a famous one – known as Iran 1996. In fact, it was proposed by Ji Chen in *Crux Mathematicorum*, april 1994.

Equality hols if and only if $x=y=z=\frac{\sqrt{2}}{\sqrt{2}}=1$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.