RMM - Geometry Marathon 1701 - 1800

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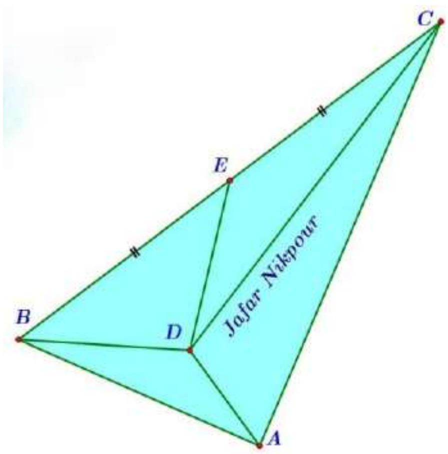
1701. Suppose that: $\angle DBA = 20^\circ; \angle DAB = 30^\circ; \angle DBC = 40^\circ; \angle DAC = 60^\circ$

Prove that: $\angle DEC = 140^\circ$



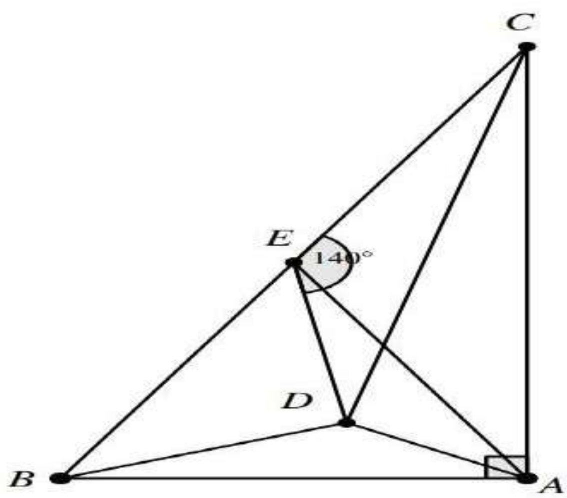
Proposed by Jafar Nikpour – Iran

Solution by Eric - Dimitrie Cismaru – Romania

We have $\sphericalangle ABC = \sphericalangle DBA + \sphericalangle DBC = 60^\circ, \sphericalangle BAC = \sphericalangle DAB + \sphericalangle DBC = 90^\circ$, so ΔABC is a right triangle and $\sphericalangle BCA = 30^\circ$.

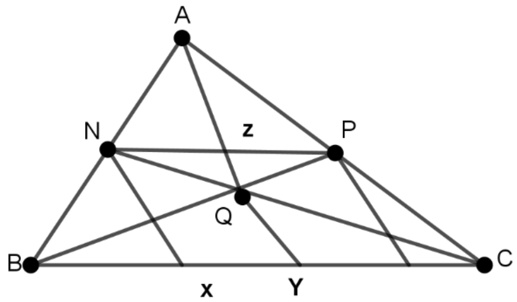
On the other hand, since E is the midpoint of BC, AE is a median in a right triangle, so $[AE] = [BE] = [EC]$, and since $\sphericalangle EBA = 60^\circ, \Delta BEA$ is equilateral, so we have $[BA] = [AE]$. The triangle ΔAEC is isosceles, so we have $\sphericalangle EAC = \sphericalangle ECA = 30^\circ$.

Therefore, $\Delta DAB \equiv \Delta DAE$, which leads us to $\sphericalangle DEA = \sphericalangle DBA = 20^\circ$, and since $\sphericalangle AEC = 120^\circ$, we obtain $\sphericalangle DEC = 140^\circ$, the conclusion.



1702. If: $NX \parallel AQ \parallel PY \parallel \Rightarrow$ Prove that:

$$\frac{XN}{NZ} \cdot \frac{ZP}{PY} = 1$$



Proposed by Romeo Cătălinoiu – Romania

Solution by Mirsadix Muzefferov – Azerbaijan

ΔABE and ΔBNX (They are similar)

Then: $\frac{XN}{AE} = \frac{BN}{BA}$ (1)

Also, ΔAEC and ΔPYC (are similar)

Then: $\frac{AE}{PY} = \frac{AC}{PC}$ (2)

Multiply (1) and (2) side by side:

$$\frac{XN}{AE} \cdot \frac{AE}{PY} = \frac{BN}{BA} \cdot \frac{AC}{PC} \Rightarrow \frac{XN}{PY} = \frac{BN}{BA} \cdot \frac{AC}{PC} \quad (3)$$

On the other hand, according to Tanasis Gakopoulos theorem, in ΔABC ...

$$\frac{NZ}{ZP} = \frac{BN}{AB} : \frac{CP}{AC} \text{ or } \frac{ZP}{NZ} = \frac{AB}{BN} \cdot \frac{CP}{AC} \quad (4)$$

Multiply (3) and (4) side by side:

$$\frac{XN}{PY} \cdot \frac{NZ}{ZP} = \left(\frac{BN}{BA} \cdot \frac{AC}{PC}\right) \cdot \left(\frac{AB}{BN} \cdot \frac{CP}{AC}\right) = 1$$

1703. In ΔABC the following relationship holds:

$$\frac{bc \cdot r_a^2}{1 + bccos(A)} + \frac{ca \cdot r_b^2}{1 + cacos(B)} + \frac{ab \cdot r_c^2}{1 + abcos(C)} \geq \frac{648r^4}{2 + 3R^2}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{bcr_a^2}{1+bccos(A)}+\frac{car_b^2}{1+cacos(B)}+\frac{abr_c^2}{1+abcos(C)}\overset{divided\ by}{=}\frac{r_a^2}{\frac{1}{bc}+cosA}+\frac{r_b^2}{\frac{1}{ac}+cosB}$$

$$+\frac{r_c^2}{\frac{1}{ab}+cosC}\geq\frac{(r_a+r_b+r_c)^2}{\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}+cosA+cosB+cosC\right)}=$$

$$=\frac{(r_a+r_b+r_c)^2}{\left(\frac{a+b+c}{abc}+1+\frac{r}{R}\right)}=\frac{(4R+r)^2}{\left(\frac{1}{2Rr}+1+\frac{r}{R}\right)}=\frac{(4R+r)^2}{\frac{1+2Rr+2r^2}{2Rr}}=\frac{(4R+r)^2\cdot 2Rr}{1+2Rr+2r^2}\overset{Euler}{\geq}$$

$$\overset{Euler}{\geq}\frac{(9r)^2\cdot 4r^2}{R^2+\frac{R^2}{2}+1}=\frac{648r^4}{2+3R^2}$$

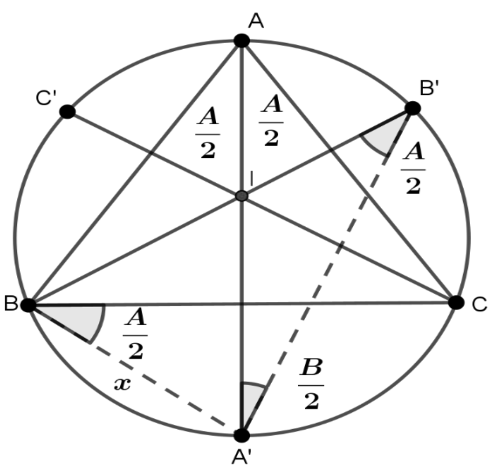
Equality holds for $a=b=c$.

1704. In $\Delta ABC, A', B', C'$ - middle points of the arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ respectively made with the circumcircle of the triangle ABC the following relationship holds:

$$\frac{6r}{R}\leq\frac{AB}{A'B'}+\frac{BC}{B'C'}+\frac{CA}{C'A'}\leq 3$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Tapas Das – India



A', B', C' are the mid points of arc BC, CA, AB so, AA', BB', CC' are the angle of bisector

From $\Delta ABA'$ we have

$$\frac{x}{\sin\frac{A}{2}} = \frac{c}{\sin C} = 2R \Rightarrow x = 2R\sin\frac{A}{2}$$

Now

$$\angle BIA' = \pi - (\angle A'BI + \angle BA'I) = \pi - \left(\frac{A}{2} + \frac{B}{2}\right) - C = \pi - \frac{A}{2} - \frac{B}{2} - [\pi - (B + A)] = \frac{A+B}{2}$$

$$\therefore \angle A'BI = \angle AIB = \frac{A+B}{2}$$

$$\therefore A'B = A'I = 2R\sin\frac{A}{2} \quad \text{(analog)}$$

From $\Delta A'IB', \angle AA'B' = \frac{B}{2}, \angle BB'A' = \frac{A}{2}$

$$\therefore \angle A'IB' = \pi - \left(\frac{A+B}{2}\right)$$

$$\therefore \sin\angle A'IB' = \sin\frac{A+B}{2} = \cos\frac{C}{2}$$

$$(\because A + B + C = \pi)$$

From $\Delta A'IB'$

$$\frac{A'B'}{\sin\angle A'IB'} = \frac{IB'}{\sin\angle IA'B'} \Rightarrow \frac{A'B'}{\cos\frac{C}{2}} = \frac{2R\sin\frac{B}{2}}{\sin\frac{B}{2}}$$

$$\therefore A'B' = 2R\cos\frac{C}{2}$$

Now

$$\frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} = 2\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \overset{Jensen}{\leq} 2 \times 3 \cdot \sin\left(\frac{A+B+C}{6}\right) =$$

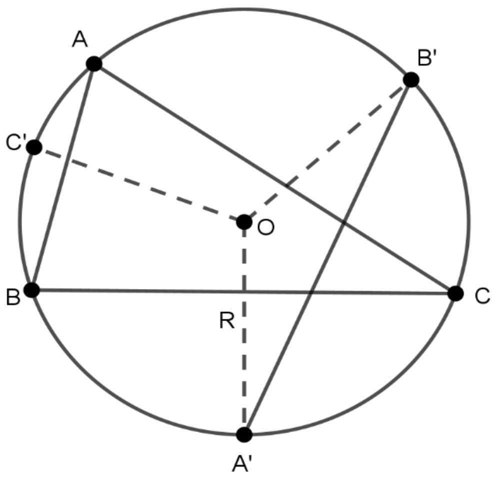
$$= 6 \cdot \sin\frac{\pi}{6} = 6 \times \frac{1}{2} = 3$$

Note $f(x) = \sin x$ is concave in $\left(0, \frac{\pi}{2}\right)$

$$\frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} = 2\left(\sum \sin\frac{A}{2}\right) \overset{AM-GM}{\geq} 6\left(\prod \sin\frac{A}{2}\right)^{\frac{1}{3}} = 6 \times \left(\frac{r}{4R}\right)^{\frac{1}{3}} =$$

$$= 6 \times \left(\frac{r^3}{4Rr^2}\right)^{\frac{1}{3}} \overset{Euler}{\geq} 6 \times \left(\frac{r^3}{4R \cdot \frac{R^2}{4}}\right) = \frac{6r}{R}$$

Solution 2 by Adrian Popa – Romania



$$\widehat{A'OB} = \widehat{A'B'} = \widehat{A'C} = \widehat{CB'} = \frac{\widehat{BC}}{2} + \frac{\widehat{AC}}{2} = \widehat{A} + \widehat{B} = \pi - \widehat{C}$$

$\Delta OA'B'$ (Cosine Theorem)

$$A'B'^2 = OA'^2 + OB'^2 - 2OA' \cdot OB' \cdot \cos \widehat{A'OB} \Rightarrow$$

$$\Rightarrow A'B'^2 = R^2 + R^2 + 2R \cdot R \cos C - 2R^2(1 + \cos C) =$$

$$= 2R^2 \cdot 2\cos^2\frac{C}{2} = 4R^2\cos^2\frac{C}{2} \Rightarrow A'B' = 2R\cos\frac{C}{2}$$

Similarly: $A'C' = 2R\cos\frac{B}{2}$ **and** $B'C' = 2R\cos\frac{A}{2}$

$$\frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} = \sum_{cyc} \frac{2R\sin C}{2R\cos\frac{C}{2}} = \sum_{cyc} \frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{C}{2}} = 2\sum_{cyc}\sin\frac{C}{2} = 2\sum_{cyc}\sin\frac{A}{2}$$

We must show that: $\frac{6r}{R} \overset{(1)}{\leq} 2\sum\sin\frac{A}{2} \overset{(2)}{\leq} 3$

(1) $$\sum\sin\frac{A}{2} \overset{MA\geq MG}{\geq} 3\sqrt[3]{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = 3\sqrt[3]{\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{(s-a)(s-c)}{ac}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}}}$$

$$= 3\sqrt[3]{\frac{S^2}{4RS\cdot s}} = 3\sqrt[3]{\frac{r}{4R}} = 3\sqrt[3]{\frac{r^3}{4Rr^2}} = \frac{3r}{\sqrt[3]{4Rr^2}} \overset{R\ge 2r}{\ge} \frac{3r}{\sqrt[3]{4R\frac{R^2}{4}}} = \frac{3r}{R}$$

$$\Rightarrow \sum \sin\frac{A}{2} \ge \frac{3r}{R} \,|\cdot 2 \Rightarrow 2\sum \sin\frac{A}{2} \ge \frac{6r}{R}$$

(2) $\sum \sin\frac{A}{2} \overset{Jensen}{\le} 3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3} = 3\sin\frac{A+B+C}{6} = 3\sin\frac{\pi}{6} = \frac{3}{2} \Rightarrow 2\sum \sin\frac{A}{2} \le 3$

1705. In ΔABC the following relationship holds:

$$\frac{\cot A \cot B \cot C}{\sin A \sin B \sin C} \le \frac{8}{27}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Note: In\ \Delta\ ABC,\ A+B+C=\pi, \sin 2A + \sin 2B + \sin 2C$$
$$= 2\sin(A+B)\cos(A-B) + \sin 2C$$
$$= 2\sin C\ [\cos(A-B) - \cos(A+B)] = 4\sin A \sin B \sin C.$$

$Case\ 1\ Let\ the\ triangle\ acute, then\ 4\sin A \sin B \sin C. \overset{AM-GM}{\ge} 3\sqrt[3]{\sin 2A.\sin 2B.\sin 2C}\ or$

$$\frac{4}{3} \ge \sqrt[3]{\frac{\sin 2A.\sin 2B.\sin 2C}{\sin^3 A.\sin^3 B \sin^3 C}} = 2\sqrt[3]{\left(\frac{\cot A \cot B \cot C}{\sin A \sin B \sin C}\right)}\ or\ \frac{2}{3} \ge \sqrt[3]{\frac{\cot A \cot B \cot C}{\sin A \sin B \sin C}}\ or$$
$$\frac{8}{27} \ge \frac{\cot A \cot B \cot C}{\sin A \sin B \sin C}$$

$Case2\ for\ non\ acute\ triangle \prod \cot A < 0\ and$

$$\prod \sin A > 0\ so\ the\ given\ expression < 0 < \frac{8}{27}, equality\ for\ A=B=C=\frac{\pi}{3}$$

1706. In ΔABC the following relationship holds:

$$cos\frac{A-B}{2}\,cos\frac{B-C}{2}\,cos\frac{C-A}{2} \ge 8sin\frac{A}{2}\,sin\frac{B}{2}\,sin\frac{C}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}\geq 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}=\frac{2\cos\frac{A-B}{2}\cos\frac{C}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}=\frac{2\cos\frac{A-B}{2}\sin\frac{A+B}{2}}{\sin C}=\frac{\sin A+\sin B}{\sin C}=$$

$$=\frac{2R\sin A+2R\sin B}{2R\sin C}=\frac{a+b}{c}\quad (1)$$

Analogously others:

$$\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\frac{b+c}{a}\quad (2)$$

$$\frac{\cos\frac{C-A}{2}}{\sin\frac{B}{2}}=\frac{a+c}{b}\quad (3)$$

If we multipy (1),(2) *and* (3) *side by side we have* ,

$$\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}\cdot\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}\cdot\frac{\cos\frac{C-A}{2}}{\sin\frac{B}{2}}=\frac{a+b}{c}\cdot\frac{b+c}{a}\cdot\frac{a+c}{b}\quad (4)$$

(4) *from here* :

$$\frac{a+b}{c}\cdot\frac{b+c}{a}\cdot\frac{a+c}{b}\overset{A-G}{\geq}\frac{2\sqrt{ab}}{c}\cdot\frac{2\sqrt{bc}}{a}\cdot\frac{2\sqrt{ac}}{b}=8\ (QED)$$

Equality holds : $a=b=c$

1707. In ΔABC the following relationship holds:

$$\frac{\sin A}{2+\cos A}+\frac{\sin B}{2+\cos B}+\frac{\sin C}{2+\cos C}\leq\frac{3\sqrt{3}}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\text{Let} f(x)=\frac{\sin x}{2+\cos x}, x\in(0,\pi).\ f'(x)=\frac{2\cos x}{(2+\cos x)^2},$$

$$f''(x)=\frac{2\sin x(\cos x-2)}{(2+\cos x)^3}<0 \text{ so } f \text{ is concave in } (0,\pi)$$

Now using this result

$$\frac{\sin A}{2+\cos A}+\frac{\sin B}{2+\cos B}+\frac{\sin C}{2+\cos C}\overset{Jensen}{\leq}3.\frac{\sin\left(\frac{\pi}{3}\right)}{2+\cos\left(\frac{\pi}{3}\right)}=\frac{3\sqrt{3}}{5}.$$

$$Equality\ for\ A=B=C=\frac{\pi}{3}$$

1708. **In any Δ ABC, the following relationship holds :**

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)\leq\left(\frac{1+\sqrt{3}}{2}\right)^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)\leq\left(\frac{1+\sqrt{3}}{2}\right)^3$$

$$\Leftrightarrow\sum_{cyc}\ln(\sin A+\cos A)\overset{(*)}{\leq}3\ln\left(\frac{1+\sqrt{3}}{2}\right)$$

Let $f(x)=\ln(\sin x+\cos x)\ \forall\ x\in(0,\pi)$ and then : $f''(x)=\frac{-2}{(\sin x+\cos x)^2}<0$

$$\Rightarrow f(x)\text{ is concave}\ \therefore\sum_{cyc}\ln(\sin A+\cos A)\overset{Jensen}{\leq}3\ln\left(\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)$$

$$=3\ln\left(\frac{1+\sqrt{3}}{2}\right)\Rightarrow(*)\text{ is true}\ \therefore(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)$$

$$\leq\left(\frac{1+\sqrt{3}}{2}\right)^3\ \forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)\overset{AM-GM}{\leq}\left(\frac{\sum\sin A+\sum\cos A}{3}\right)^3=$$

$$=\left(\frac{\frac{s}{R}+\frac{r}{R}+1}{3}\right)^3\overset{MITRINOVIC}{\leq}\left(\frac{\frac{3\sqrt{3}R}{2R}+\frac{r}{R}+1}{3}\right)^3\overset{EULER}{\leq}\left(\frac{\frac{3\sqrt{3}}{2}+\frac{1}{2}+1}{3}\right)^3=\left(\frac{\sqrt{3}+1}{2}\right)^3$$

Equality holds for $a=b=c$.

1709. In any Δ ABC, the following relationship holds :

$$\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}\ge\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnem

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{LHS}=\sum_{\text{cyc}}\frac{a^2}{a.\sqrt{a^2+3bc}}\overset{\text{Bergstrom}}{\ge}\frac{4s^2}{\sum_{\text{cyc}}\sqrt{a}.\sqrt{a^3+3abc}}\overset{\text{CBS}}{\ge}$$

$$\frac{4s^2}{\sqrt{2s}.\sqrt{2s(s^2-6Rr-3r^2)+36Rrs}}\overset{?}{\ge}\frac{3}{2}\Leftrightarrow 7s^2\underset{(*)}{\overset{?}{\ge}}108Rr-27r^2$$

$$\text{Now}, 7s^2\overset{\text{Gerretsen}}{\ge}112Rr-35r^2=108Rr-27r^2+4r(R-2r)\overset{\text{Euler}}{\ge}108Rr-27r^2$$

$$\Rightarrow(*)\text{ is true }\therefore\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}\ge\frac{3}{2}$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)

Solution 2 by Tapas Das-India

$$\text{In any } \Delta ABC \text{ prove that }\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}\ge\frac{3}{2}\ (\text{Nguyen Hung Cuong})$$

$$\text{solution}:\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}$$

$$=\sum\frac{a^{\frac{3}{2}}}{\sqrt{a^3+3abc}}\ge\frac{(a+b+c)^{\frac{3}{2}}}{\sqrt{a^3+b^3+c^3+9abc}}(\text{Radon})$$

$$=\frac{(2s)^{\frac{3}{2}}}{\sqrt{(2s)(s^2+12Rr-3r^2)}}$$

$$=\frac{2s}{\sqrt{s^2+12Rr-3r^2}}\text{, now we need to show}$$

$$\frac{2s}{\sqrt{s^2+12Rr-3r^2}}\ge\frac{3}{2}\text{ or}$$

$$7s^2\ge 108Rr-27r^2\text{ or }R\overset{\text{Gerretsen}}{\ge}2r\ (\text{Euler})$$

1710. In ΔABC the following relationship holds:

$$\cos A\cos B\cos C\le\frac{r^2}{2R^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$cosAcosBcosC = \frac{s^2-(2R+r)^2}{4R^2} \overset{GERRETSEN}{\leq}$$

$$\leq \frac{4R^2+4Rr+3r^2-(2R+r)^2}{4R^2} = \frac{2r^2}{4R^2} = \frac{r^2}{2R^2}$$

Equality holds for $a=b=c$.

1711. In ΔABC the following relationship holds:

$$atan\frac{A}{2}+btan\frac{B}{2}+ctan\frac{C}{2} \geq 6r$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $a \leq b \leq c \Rightarrow A \leq B \leq C \Rightarrow tan\frac{A}{2} \leq tan\frac{B}{2} \leq tan\frac{C}{2}$

$$atan\frac{A}{2}+btan\frac{B}{2}+ctan\frac{C}{2} \overset{CEBYSHEV}{\geq} \frac{1}{3}(a+b+c)\sum_{cyc} tan\frac{A}{2} =$$

$$= \frac{2s}{3}\sum_{cyc} tan\frac{A}{2} \overset{MITRINOVIC}{\geq} \frac{2\cdot 3\sqrt{3}r}{3}\sum_{cyc} tan\frac{A}{2} \overset{JENSEN}{\geq}$$

$$\geq 2\sqrt{3}r\cdot 3tan\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right) = 6\sqrt{3}r\cdot tan\frac{A+B+C}{6} =$$

$$= 6\sqrt{3}r\cdot tan\frac{\pi}{6} = 6\sqrt{3}r\cdot\frac{\sqrt{3}}{3} = 6r$$

Equality holds for $a=b=c$.

1712.

In any Δ ABC, the following relationship holds :

$$\frac{l_a+l_b+l_c}{R} \leq 2\left(\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2\left(\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\right)=2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\sum_{cyc}\frac{1}{\cos\frac{A}{2}}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{2s}{4R}.\frac{9}{\sum_{cyc}\cos\frac{A}{2}}\overset{\text{Jensen}}{\geq}\frac{2s}{4R}.\frac{9}{3.\frac{\sqrt{3}}{2}}$$

$$\left(\because f(x)=\cos\frac{x}{2}\ \forall x\in(0,\pi)\Rightarrow f''(x)=\frac{-\cos\frac{x}{2}}{4}<0\Rightarrow f(x)\text{ is concave}\right)=\frac{\sqrt{3}s}{R}$$

$$\geq\frac{l_a+l_b+l_c}{R}\left(\begin{array}{c}\because l_a\leq\sqrt{s(s-a)}\text{ and analogs}\Rightarrow\sum_{cyc}l_a\overset{CBS}{\leq}\\ \sqrt{3}.\sqrt{\sum_{cyc}s(s-a)}=\sqrt{3}s\end{array}\right)$$

$$\therefore\frac{l_a+l_b+l_c}{R}\leq2\left(\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\right)$$

$$\forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\text{ is equilateral (QED)}$$

1713. In ΔABC the following relationship holds:

$$F\leq\frac{1}{\sqrt{3}}\sqrt[3]{(m_a^2m_b^2m_c^2)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma: In ΔABC the following relationship holds:

$$m_a\geq\sqrt{s(s-a)}$$

Proof (by editor):

$$m_a\geq\sqrt{s(s-a)}\Leftrightarrow m_a^2\geq\frac{a+b+c}{2}\cdot\frac{b+c-a}{2}\Leftrightarrow$$

$$\frac{b^2+c^2}{2}-\frac{a^2}{4}\geq\frac{b^2+c^2+2bc-a^2}{4}\Leftrightarrow2b^2+2c^2\geq b^2+c^2+2bc$$

$$b^2+c^2-2bc\geq0\Leftrightarrow(b-c)^2\geq0$$

Back to the problem:
By lemma:

$$\prod m_a^2 \geq s^3(s-a)(s-b)(s-c) = s^4r^2 \overset{Mitrinovic}{\geq} s^3.3\sqrt{3}\,r^3$$

$$\frac{1}{\sqrt{3}}\sqrt[3]{(m_a^2m_b^2m_c^2)} \geq \frac{1}{\sqrt{3}}\sqrt[3]{(s^3.3\sqrt{3}\,r^3)} = r.s = F$$

Equality holds for $a = b = c$.

1714. In ΔABC the following relationship holds:

$$a^2\cos\frac{A}{2} + b^2\cos\frac{B}{2} + c^2\cos\frac{C}{2} \leq \frac{9\sqrt{3}}{2}R^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\cos\frac{A}{2} \overset{JENSEN}{\leq} 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$$

$$.\,a^2\cos\frac{A}{2} + b^2\cos\frac{B}{2} + c^2\cos\frac{C}{2} \overset{CEBYSHEV}{\leq} \frac{1}{3}\left(\sum a^2\right)\left(\sum\cos\frac{A}{2}\right) \leq$$

$$\overset{LEIBNIZ}{\leq} \frac{1}{3}9R^2\frac{3\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}R^2$$

Equality holds for $a = b = c$.

1715. In ΔABC the following relationship holds:

$$cos^2\left(\frac{A-B}{2}\right) + cos^2\left(\frac{B-C}{2}\right) + cos^2\left(\frac{C-A}{2}\right) \geq 24sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Lemma 1: In ΔABC the following relationship holds:

$$cos^2\left(\frac{A-B}{2}\right) + cos^2\left(\frac{B-C}{2}\right) + cos^2\left(\frac{C-A}{2}\right) = \frac{s^2+r^2+2Rr}{4R^2} + 1$$

Proof:

$$cos^2\left(\frac{A-B}{2}\right) + cos^2\left(\frac{B-C}{2}\right) + cos^2\left(\frac{C-A}{2}\right) = \sum_{cyc} cos^2\left(\frac{A-B}{2}\right) =$$

$$=\sum_{cyc}\frac{1+cos(A-B)}{2}=\frac{3}{2}+\frac{1}{2}\sum_{cyc}cosAcosB+\frac{1}{2}\sum_{cyc}sinAsinB=$$

$$=\frac{3}{2}+\frac{s^2+r^2-4R^2}{8R^2}+\frac{1}{2}\sum_{cyc}\frac{ab}{4R^2}=$$

$$=\frac{12R^2+s^2+r^2-4R^2+s^2+r^2+4Rr}{8R^2}=\frac{s^2+r^2+2Rr}{4R^2}+1$$

Lemma 2: In ΔABC the following relationship holds:

$$sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=\frac{r}{4R}$$

Proof:

$$sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=\prod_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}}=$$

$$=\frac{(s-a)(s-b)(s-c)}{abc}=\frac{s(s-a)(s-b)(s-c)}{sabc}=\frac{F^2}{s\cdot 4RF}=$$

$$=\frac{F}{4Rs}=\frac{rs}{4Rs}=\frac{r}{4R}$$

Using Lemma 1 and Lemma 2 we must prove that:

$$\frac{s^2+r^2+2Rr}{4R^2}+1\geq 24\cdot\frac{r}{4R}$$

$$s^2+r^2+2Rr+4R^2\geq 24Rr$$

$$s^2\geq 22Rr-r^2-4R^2 \text{ (to prove)}$$

$$s^2\overset{GERRETSEN}{\geq}16Rr-5r^2\geq 22Rr-r^2-4R^2\Leftrightarrow$$

$$\Leftrightarrow 4R^2-6Rr-4r^2\geq 0\Leftrightarrow 2R^2-3Rr-2r^2\geq 0$$

$$2R^2-4Rr+Rr-2r^2\geq 0$$

$$2R(R-2r)+r(R-2r)\geq 0$$

$$(R-2r)(2R+r)\geq 0$$

$$R-2r\geq 0$$
$$R\geq 2r\ (Euler)$$

Equality holds for $a = b = c$.

1716. **In ΔABC the following relationship holds:**

$$\cos A + \cos B + \cos C + \frac{4}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \ge \frac{67}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\cos A + \cos B + \cos C + \frac{4}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} =$$

$$= 1 + \frac{r}{R} + \frac{4}{\frac{r}{4R}} = 1 + \frac{1}{x} + 16x\left(where\,\frac{R}{r} = x \ge 2\ Euler\right)$$

we need to show:

$$1 + \frac{1}{x} + 16x \ge \frac{67}{2}\ or$$

$$32x^2 - 65x + 2 \ge 0\ or$$

$$(32x - 1)(x - 2) \ge 0\ true\ as\ x \ge 2$$

Equality holds for $a = b = c$.

1717. **In any Δ ABC, the following relationship holds:**

$$\sum_{cyc}\sqrt[3]{\frac{a}{b+c}} + \frac{R^{2024}}{r^{2024}} \ge 2^{2024} + \sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc} a^2}{\sum_{cyc} ab} \overset{?}{\le} \frac{R}{2r} \Leftrightarrow R(s^2 + 4Rr + r^2) \overset{?}{\ge} 4r(s^2 - 4Rr - r^2)$$

$$\Leftrightarrow (R - 4r)s^2 + (R + 4r)(4Rr + r^2) \underset{(*)}{\overset{?}{\ge}} 0$$

$$\text{Now, LHS of } (*) = (R - 2r)s^2 - 2rs^2 + (R + 4r)(4Rr + r^2) \overset{\text{Gerretsen}}{\ge}$$

$$(R - 2r)(16Rr - 5r^2) - 2r(4R^2 + 4Rr + 3r^2) + (R + 4r)(4Rr + r^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow 3R^2 - 7Rr + 2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(3R-r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\therefore \frac{\sum_{cyc} a^2}{\sum_{cyc} ab} \leq \frac{R}{2r} \to (1)$$

$$\text{Now, } \sum_{cyc} \sqrt[3]{\frac{a}{b+c}} = \frac{1}{\sqrt[3]{2}} \sum_{cyc} \sqrt[3]{\frac{2a}{b+c}.1.1} \overset{G-H}{\geq} \frac{3}{\sqrt[3]{2}} \sum_{cyc} \frac{\frac{2a}{b+c}}{\frac{2a}{b+c} + \frac{2a}{b+c} + 1}$$

$$= \frac{6}{\sqrt[3]{2}} \sum_{cyc} \frac{a}{4a+b+c} = \frac{6}{\sqrt[3]{2}} \sum_{cyc} \frac{a^2}{4a^2+ab+ca} \overset{\text{Bergstrom}}{\geq} \frac{6}{\sqrt[3]{2}} \cdot \frac{\left(\sum_{cyc} a\right)^2}{4\sum_{cyc} a^2 + 2\sum_{cyc} ab}$$

$$= \frac{6}{\sqrt[3]{2}} \cdot \frac{u+2v}{4u+2v} \left(u = \sum_{cyc} a^2, v = \sum_{cyc} ab \right) \overset{?}{\geq} \frac{3}{\sqrt[3]{2}} \cdot \frac{v}{u}$$

$$\Leftrightarrow 2u^2 + 4uv \overset{?}{\geq} 4uv + 2v^2 \Leftrightarrow u \overset{?}{\geq} v \to \text{true} \therefore \sum_{cyc} \sqrt[3]{\frac{a}{b+c}} \geq \frac{3}{\sqrt[3]{2}} \cdot \frac{\sum_{cyc} ab}{\sum_{cyc} a^2}$$

$$\overset{\text{via }(1)}{\geq} \frac{3}{\sqrt[3]{2}} \cdot \frac{2r}{R} \therefore \sum_{cyc} \sqrt[3]{\frac{a}{b+c}} \geq \frac{3}{\sqrt[3]{2}} \cdot \frac{2r}{R} \to (2)$$

$$\text{We have}: r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}} \right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Again, } \sum_{cyc} \sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}}} = \frac{1}{\sqrt[3]{2}} \sum_{cyc} \sqrt[3]{\frac{2r_a}{r_b+r_c}.1.1} \overset{A-G}{\leq} \frac{1}{3.\sqrt[3]{2}} \sum_{cyc} \left(\frac{2r_a}{r_b+r_c} + 2 \right)$$

$$= \frac{2}{3.\sqrt[3]{2}} \cdot \left(\sum_{cyc} r_a \right) \left(\sum_{cyc} \frac{1}{r_b+r_c} \right) \overset{\text{via (i)}}{=} \frac{2}{3.\sqrt[3]{2}} . (4R+r) . \sum_{cyc} \frac{1}{4R\cos^2\frac{A}{2}}$$

$$= \frac{2}{3.\sqrt[3]{2}} \cdot \left(\frac{4R+r}{4R} \right) \cdot \left(1 + \frac{(4R+r)^2}{s^2} \right) \overset{\substack{\text{Euler} \\ \text{and} \\ \text{Mitrinovic}}}{\leq} \frac{2}{3.\sqrt[3]{2}} \cdot \left(\frac{\frac{9R}{2}}{4R} \right) \cdot \left(1 + \frac{\frac{81R^2}{4}}{27r^2} \right)$$

$$\Rightarrow \sum_{cyc} \sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}}} \leq \frac{3}{4.\sqrt[3]{2}} \cdot \left(1 + \frac{3R^2}{4r^2} \right) \to (3)$$

$$\text{Let } F(n) = t^n - 2^n \ \forall\, t = \frac{R}{r} \overset{\text{Euler}}{\geq} 2 \ (t \to \text{fixed}) \text{ and } \forall\, n \geq 2 \text{ and then}:$$

$$F'(n) = t^n.(\ln t) - 2^n.(\ln 2) \geq 0$$

$$(\because t^n \geq 2^n \text{ and } \ln t \geq \ln 2 \Rightarrow t^n.(\ln t) - 2^n.(\ln 2) \geq 0)$$

$$\therefore F(n) \text{ is } \uparrow \forall\, n \geq 2 \Rightarrow F(n) \geq F(2) \Rightarrow \text{when both t and n vary, } \left(\frac{R}{r}\right)^n - 2^n$$

$$\geq \frac{R^2}{r^2} - 4 \Rightarrow \frac{R^{2024}}{r^{2024}} - 2^{2024} \geq \frac{R^2}{r^2} - 4 \to (4)$$

$$\text{So, } \sum_{cyc}\sqrt[3]{\frac{a}{b+c}}+\frac{R^{2024}}{r^{2024}}-2^{2024}-\sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}}\overset{\text{via }(2),(3)\text{ and }(4)}{\geq}$$

$$\frac{3}{\sqrt[3]{2}}\cdot\frac{2r}{R}+\frac{R^2}{r^2}-4-\frac{3}{4.\sqrt[3]{2}}\cdot\left(1+\frac{3R^2}{4r^2}\right)=\frac{R^2-4r^2}{r^2}-\frac{3}{\sqrt[3]{2}}\cdot\left(\frac{1}{4}+\frac{3R^2}{16r^2}-\frac{2r}{R}\right)$$

$$=\frac{R^2-4r^2}{r^2}-\frac{3}{16.\sqrt[3]{2}}\cdot\left(\frac{3R^3+4Rr^2-32r^3}{Rr^2}\right)\geq$$

$$\frac{R^2-4r^2}{r^2}-\frac{1}{6}\cdot\left(\frac{3R^3+4Rr^2-32r^3}{Rr^2}\right)$$

$$\left(\because\frac{3}{16.\sqrt[3]{2}}\approx 0.1488<\frac{1}{6}\text{ and }3R^3+4Rr^2-32r^3\overset{\text{Euler}}{\geq}24r^3+8r^3-32r^3=0\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 3t^3-28t+32\overset{?}{\geq}0\Leftrightarrow(t-2)\big((t-2)(3t+12)+8\big)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow\sum_{cyc}\sqrt[3]{\frac{a}{b+c}}+\frac{R^{2024}}{r^{2024}}-2^{2024}-\sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}}\geq 0$$

$$\therefore\sum_{cyc}\sqrt[3]{\frac{a}{b+c}}+\frac{R^{2024}}{r^{2024}}\geq 2^{2024}+\sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}}$$

$\forall\,\Delta\,ABC$, " $=$ " iff $\Delta\,ABC$ is equilateral (QED)

1718. In ΔABC the following relationship holds:

$$\min\left(\sum_{cyc}(b+c)h_a,\sum_{cyc}a(h_b+h_c)\right)\geq 36\sqrt{3}r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}(b+c)h_a=\sum_{cyc}(b+c)\cdot\frac{2F}{a}=2F\sum_{cyc}\frac{b+c}{a}\overset{AM-GM}{\geq}$$

$$\geq 2F\cdot 3\sqrt[3]{\prod_{cyc}\frac{b+c}{a}}=6rs\cdot\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{abc}}\overset{CESARO}{\geq}$$

$$\geq 6rs\cdot\sqrt[3]{\frac{8abc}{abc}}=6rs\sqrt[3]{8}=12rs\overset{MITRINOVIC}{\geq}12r\cdot 3\sqrt{3}r=36\sqrt{3}r^2$$

$$\sum_{cyc} a(h_b+h_c)=\sum_{cyc} a\left(\frac{2F}{b}+\frac{2F}{c}\right)=2F\sum_{cyc} a\left(\frac{1}{b}+\frac{1}{c}\right)=$$

$$=2rs\sum_{cyc}\frac{a(b+c)}{bc}\overset{AM-GM}{\geq}2rs\cdot 3\sqrt[3]{\frac{abc(b+c)(c+a)(a+b)}{abc\cdot abc}}=$$

$$=6rs\cdot\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{abc}}\overset{CESARO}{\geq}$$

$$\geq 6rs\cdot\sqrt[3]{\frac{8abc}{abc}}=6rs\sqrt[3]{8}=12rs\overset{MITRINOVIC}{\geq}12r\cdot 3\sqrt{3}r=36\sqrt{3}r^2$$

Equality holds for $a=b=c$.

1719. **In ΔABC the following relationship holds:**

$$\frac{AI\cdot II_a}{b+c}+\frac{BI\cdot II_b}{c+a}+\frac{CI\cdot II_c}{a+b}\geq 2\sqrt{3}r$$

Proposed by Ertan Yildirim-Turkiye

Solution by Daniel Sitaru-Romania

$$AI=\frac{r}{\sin\frac{A}{2}}\quad BI=\frac{r}{\sin\frac{B}{2}}\quad CI=\frac{r}{\sin\frac{C}{2}}$$

$$II_a=4R\sin\frac{A}{2}\quad II_b=4R\sin\frac{B}{2}\quad II_c=4R\sin\frac{C}{2}$$

$$\frac{AI\cdot II_a}{b+c}+\frac{BI\cdot II_b}{c+a}+\frac{CI\cdot II_c}{a+b}=\sum_{cyc}\frac{AI\cdot II_a}{b+c}=\sum_{cyc}\frac{\frac{r}{\sin\frac{A}{2}}\cdot 4R\sin\frac{A}{2}}{b+c}=$$

$$=4Rr\sum_{cyc}\frac{1}{b+c}\overset{BERGSTROM}{\geq}4Rr\cdot\frac{(1+1+1)^2}{b+c+c+a+a+b}=$$

$$=4Rr\cdot\frac{9}{4s}=Rr\cdot\frac{9}{s}\overset{MITRINOVIC}{\geq}Rr\cdot\frac{9}{\frac{3\sqrt{3}}{2}R}=\frac{6r}{\sqrt{3}}=2\sqrt{3}r$$

Equality holds for $a=b=c$.

1720. **In ΔABC the following relationship holds:**

$$\frac{m_a+m_b+m_c}{R^2}\le\frac{1}{r}\sum\cos\frac{B}{2}\cos\frac{C}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{m_a+m_b+m_c}{R^2}\overset{Leunberger}{\le}\frac{4R+r}{R^2}\overset{Euler}{\le}\frac{9R}{2R^2}=\frac{9}{2R}\quad(1)$$

$$\frac{1}{r}\sum\cos\frac{B}{2}\cos\frac{C}{2}\overset{AM-GM}{\ge}\frac{3}{r}\sqrt[3]{\prod\cos^2\frac{A}{2}}=\frac{3}{r}\sqrt[3]{\left(\frac{s^2}{16R^2}\right)}\ge$$

$$\ge\frac{3}{r}\sqrt[3]{\left(\frac{s^3}{16R^2s}\right)}\overset{Mitrinovic}{\ge}\frac{3}{r}\sqrt[3]{\frac{s^3}{8R^3 3\sqrt{3}}}=\sqrt{3}\frac{s}{2Rr}\overset{Mitrinovic}{\ge}\frac{9}{2R}\qquad(2)$$

$$from(1)\ \&\ (2)we\ get\ In\ \Delta\ ABC:\frac{m_a+m_b+m_c}{R^2}\le\frac{1}{r}\sum\cos\frac{B}{2}\cos\frac{C}{2}$$

1721. **G –centroid of $\Delta ABC, A', B', C'\in Ext(\Delta ABC), (G,A,A'), (G,B,B'), (G,C,C')$ –collinears, $AA'=BC, BB'=CA, CC'=AB$. Prove that:**

$$\frac{[A'B'C']}{[ABC]}\ge\left(1+\frac{2r}{R}\right)^2$$

Proposed by Mehmet Şahin-Turkiye

Solution by Daniel Sitaru-Romania

$$GA=\frac{2}{3}m_a, GB=\frac{2}{3}m_b, GC=\frac{2}{3}m_c$$

$$cos(\measuredangle AGB)=\frac{\left(\frac{2}{3}m_a\right)^2+\left(\frac{2}{3}m_b\right)^2-c^2}{2\cdot\frac{2}{3}m_a\cdot\frac{2}{3}m_b}$$

$$GA'=\frac{4}{3}m_a, GB'=\frac{4}{3}m_b, GC'=\frac{4}{3}m_c$$

$$A'B'^2=\left(\frac{4}{3}m_a\right)^2+\left(\frac{4}{3}m_b\right)^2-2\cdot\frac{4}{3}m_a\cdot\frac{4}{3}m_b\cdot cos(\measuredangle AGB)$$

$$A'B'^2 = \left(\frac{4}{3}m_a\right)^2 + \left(\frac{4}{3}m_b\right)^2 - 2\cdot\frac{4}{3}m_a\cdot\frac{4}{3}m_b\cdot\frac{\left(\frac{2}{3}m_a\right)^2+\left(\frac{2}{3}m_b\right)^2-c^2}{2\cdot\frac{2}{3}m_a\cdot\frac{2}{3}m_b}$$

$$A'B'^2 = \left(\frac{4}{3}m_a\right)^2 + \left(\frac{4}{3}m_b\right)^2 - 4\left(\left(\frac{2}{3}m_a\right)^2+\left(\frac{2}{3}m_b\right)^2\right)+4c^2$$

$$A'B'^2 = 4c^2 \Rightarrow A'B' = 2c, B'C' = 2a, C'A' = 2b$$

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = 2 \Rightarrow \frac{[A'B'C']}{[ABC]} = 2^2 = 4$$

$$4 \ge \left(1+\frac{2r}{R}\right)^2 \Leftrightarrow 2 \ge 1+\frac{2r}{R} \Leftrightarrow R \ge 2r \text{ (Euler)}$$

Equality holds for $a = b = c$.

1722. In ΔABC the following relationship holds:

$$(m_a^2+m_b^2)\cos C + (m_b^2+m_c^2)\cos A + (m_c^2+m_a^2)\cos B \le \frac{27R^2}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$m_a^2 + m_b^2 = \frac{a^2+b^2+4c^2}{4} = \frac{(a^2+b^2+c^2)+3c^2}{4}$$

$$WLOG\ a \ge b \ge c, \cos A \le \cos B \le \cos C$$

$$(m_a^2+m_b^2)\cos C + (m_b^2+m_c^2)\cos A + (m_c^2+m_a^2)\cos B =$$

$$= \sum \frac{(a^2+b^2+c^2)+3c^2}{4}\cos C =$$

$$= \frac{1}{4}\left(\sum a^2\right)\left(\sum \cos C\right) + \frac{3}{4}\left(\sum c^2 \cos C\right) \overset{CEBYSHEV}{\le}$$

$$\le \frac{1}{4}\left(\sum a^2\right)\left(\sum \cos C\right) + \frac{3}{4}\frac{1}{3}\left(\sum a^2\right)\left(\sum \cos C\right) =$$

$$= \frac{1}{2}\left(\sum a^2\right)\left(\sum \cos C\right) \overset{LEIBNIZ}{\le} \frac{1}{2}\cdot 9R^2\left(1+\frac{r}{R}\right) \le$$

$$\leq \frac{9R^2}{2}\left(1+\frac{R}{2R}\right)=\frac{27R^2}{4}(Euler)$$

Equality holds for: $a=b=c$.

1723. In ΔABC the following relationship holds:

$$\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\leq\frac{5}{8}+\frac{r}{4R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum ab=s^2+r^2+4Rr\overset{Gerretsen}{\leq}4(R+r)^2\quad(1)$$

$$\sum\frac{1}{\sin\frac{A}{2}}=\sum\sqrt{\frac{bc}{(s-b)(s-c)}}\overset{CBS}{\leq}\sqrt{ab+bc+ca}.\sqrt{\sum\frac{1}{(s-b)(s-c)}}\overset{(1)}{\leq}2(R+r).\frac{1}{r}$$

$$\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}=\prod\sin\frac{A}{2}.\sum\frac{1}{\sin\frac{A}{2}}\leq$$

$$\leq\frac{r}{4R}.\frac{2(R+r)}{r}=\frac{1}{2}+\frac{r}{2R}=\frac{5}{8}-\frac{1}{8}+\frac{r}{2R}=$$

$$=\frac{5}{8}-\frac{1}{4}.\frac{1}{2}+\frac{r}{2R}\leq\frac{5}{8}+\frac{r}{2R}-\frac{r}{4R}(Euler)=\frac{5}{8}+\frac{r}{4R}$$

1724. In ΔABC the following relationship holds:

$$\frac{sinAsinB}{\sin^2\frac{C}{2}}+\frac{sinBsinC}{\sin^2\frac{A}{2}}+\frac{sinCsinA}{\sin^2\frac{B}{2}}\geq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{sinAsinB}{\sin^2\frac{C}{2}}+\frac{sinBsinC}{\sin^2\frac{A}{2}}+\frac{sinCsinA}{\sin^2\frac{B}{2}}=$$

$$= \frac{1}{4R^2}\sum \frac{a^2b^2}{(s-a)(s-b)} = \frac{1}{4R^2.sr^2}\sum a^2b^2(s-c)$$

$$= \frac{1}{4R^2r^2s}\left[s\sum a^2b^2 - abc\sum ab\right] \overset{\sum x^2 \ge \sum xy}{\ge}$$

$$\ge \frac{1}{4R^2r^2s}\left[s.abc\sum a - abc\sum ab\right] \ge \frac{1}{4R^2r^2s}abc[2s^2 - s^2 - r^2 - 4Rr] \overset{Gerretsen}{\ge}$$

$$\ge \frac{4Rrs}{4R^2r^2s}[12Rr - 6r^2] = 12 - \frac{6r}{R} \ge 12 - 6.\frac{1}{2}(Euler) = 9$$

Equality holds for $a = b = c$.

1725. In ΔABC the following relationship holds:

$$m_a sinA + m_b sinB + m_c sinC \le \frac{9\sqrt{3}R}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $a \le b \le c$

$$a \le b \le c \Rightarrow A \le B \le C \Rightarrow sinA \le sinB \le sinC$$

$$a \le b \le c \Rightarrow m_a \ge m_b \ge m_c$$

$$\sum_{cyc} m_a sinA \le \frac{1}{3}\cdot\sum_{cyc} m_a \cdot \sum_{cyc} sinA = \frac{1}{3}\cdot\frac{s}{R}\cdot\sum_{cyc} m_a \le$$

$$\overset{GOTMAN}{\le} \frac{s}{3R}\cdot\frac{9R}{2} = \frac{3s}{2} \overset{MITRINOVIC}{\le} \frac{3}{2}\cdot\frac{3\sqrt{3}}{2}\cdot R = \frac{9\sqrt{3}R}{4}$$

Equality holds for $a = b = c$.

1726. In ΔABC the following relationship holds:

$$\left(1 + 9\tan^2\frac{A}{2}\right)\left(1 + 9\tan^2\frac{B}{2}\right)\left(1 + 9\tan^2\frac{C}{2}\right) \ge 64$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(1+9\tan^2\frac{A}{2}\right)\left(1+9\tan^2\frac{B}{2}\right)\left(1+9\tan^2\frac{C}{2}\right)\geq 64$$

$$1+9\sum\tan^2\frac{A}{2}+81\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}+729\prod\tan^2\frac{A}{2}\geq 64$$

$$1+9\left[\left(\frac{4R+r}{s}\right)^2-2\right]+81\left[\frac{s^2-2r^2-8Rr}{s^2}\right]+729\frac{r^2}{s^2}\geq 64$$

$$9(4R+r)^2-648Rr-162r^2+729r^2\geq 0$$

$$(4R+r)^2-72Rr+63r^2\geq 0$$

$$\left(\frac{R}{r}-2\right)^2\geq 0$$

$$\textit{Equality holds for } A=B=C=\frac{\pi}{3}$$

1727. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{a^4}{a^4+2b^4}+\frac{R^{2024}}{r^{2024}}\geq 2^{2024}+\sum_{cyc}\frac{b^4}{a^4+2c^4}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b^4}{a^4+2c^4}=\frac{b^4}{a^4+2c^4}+\frac{c^4}{b^4+2a^4}+\frac{a^4}{c^4+2b^4}\overset{\text{Reverse Bergstrom}}{\leq}$$

$$\frac{b^4}{9}\left(\frac{1}{a^4}+\frac{2}{c^4}\right)+\frac{c^4}{9}\left(\frac{1}{b^4}+\frac{2}{a^4}\right)+\frac{a^4}{9}\left(\frac{1}{c^4}+\frac{2}{b^4}\right)$$

$$\Rightarrow\sum_{cyc}\frac{b^4}{a^4+2c^4}\leq\frac{2}{9}\sum_{cyc}\frac{a^4}{b^4}+\frac{1}{9}\sum_{cyc}\frac{b^4}{a^4}\to(1)$$

$$\sum_{cyc}\frac{2b^4}{a^4+2b^4}=\frac{2b^4}{a^4+2b^4}+\frac{2c^4}{b^4+2c^4}+\frac{2a^4}{c^4+2a^4}\overset{\text{Reverse Bergstrom}}{\leq}$$

$$\frac{2b^4}{9}\left(\frac{1}{a^4}+\frac{2}{b^4}\right)+\frac{2c^4}{9}\left(\frac{1}{b^4}+\frac{2}{c^4}\right)+\frac{2a^4}{9}\left(\frac{1}{c^4}+\frac{2}{a^4}\right)$$

$$\Rightarrow \sum_{cyc}\frac{2b^4}{a^4+2b^4}\le\frac{2}{9}\sum_{cyc}\frac{b^4}{a^4}+\frac{4}{3}\to(2)$$

$$\therefore \sum_{cyc}\frac{b^4}{a^4+2c^4}-\sum_{cyc}\frac{a^4}{a^4+2b^4}=\sum_{cyc}\frac{b^4}{a^4+2c^4}-\sum_{cyc}\frac{a^4+2b^4-2b^4}{a^4+2b^4}$$

$$=-3+\sum_{cyc}\frac{b^4}{a^4+2c^4}+\sum_{cyc}\frac{2b^4}{a^4+2b^4}\overset{via\ (1)\ and\ (2)}{\le}$$

$$-3+\frac{2}{9}\sum_{cyc}\frac{a^4}{b^4}+\frac{1}{9}\sum_{cyc}\frac{b^4}{a^4}+\frac{2}{9}\sum_{cyc}\frac{b^4}{a^4}+\frac{4}{3}$$

$$=-3+\frac{4}{3}+\frac{1}{3}\left(\sum_{cyc}\frac{a^4}{b^4}+\sum_{cyc}\frac{b^4}{a^4}\right)-\frac{1}{9}\sum_{cyc}\frac{a^4}{b^4}\overset{A-G}{\le}$$

$$3+\frac{4}{3}+\frac{1}{3}\sum_{cyc}\left(\frac{b^4}{c^4}+\frac{c^4}{b^4}\right)-\frac{1}{9}.3\sqrt[3]{\frac{a^4}{b^4}.\frac{b^4}{c^4}.\frac{c^4}{a^4}}=-2+\frac{1}{3}\sum_{cyc}\left(\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)^2-2\right)$$

$$=-2+\frac{1}{3}\sum_{cyc}\left(\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2-2\right)^2-2\right)\overset{Bandila}{\le}-2+\frac{1}{3}\sum_{cyc}\left(\left(\frac{R^2}{r^2}-2\right)^2-2\right)$$

$$=-2+\frac{1}{3}\sum_{cyc}\left(\frac{R^4}{r^4}-\frac{4R^2}{r^2}+2\right)=-2+\frac{1}{3}\left(\frac{3R^4}{r^4}-\frac{12R^2}{r^2}+6\right)$$

$$\therefore \sum_{cyc}\frac{b^4}{a^4+2c^4}-\sum_{cyc}\frac{a^4}{a^4+2b^4}\le\frac{R^4}{r^4}-\frac{4R^2}{r^2}\to(3)$$

Let $F(n)=t^n-2^n\ \forall\ t=\frac{R}{r}\overset{Euler}{\ge}2\ (t\to fixed)$ and $\forall\ n\ge 2$ and then :

$$F'(n)=t^n.(\ln t)-2^n.(\ln 2)\ge 0$$

$$(\because t^n\ge 2^n \text{ and } \ln t\ge\ln 2\Rightarrow t^n.(\ln t)-2^n.(\ln 2)\ge 0)$$

$\therefore$ for n and t both being variables, $F(n)$ is $\uparrow\ \forall\ n\ge 2\Rightarrow F(2024)\ge F(4)$

$$\Rightarrow t^{2024}-2^{2024}\ge t^4-16\Rightarrow\frac{R^{2024}}{r^{2024}}-2^{2024}\ge\frac{R^4}{r^4}-16\overset{Euler}{\ge}\frac{R^4}{r^4}-\frac{4R^2}{r^2}$$

$$\overset{via\ (3)}{\ge}\sum_{cyc}\frac{b^4}{a^4+2c^4}-\sum_{cyc}\frac{a^4}{a^4+2b^4}\Rightarrow\sum_{cyc}\frac{a^4}{a^4+2b^4}+\frac{R^{2024}}{r^{2024}}$$

$$\geq 2^{2024}+\sum_{cyc}\frac{b^4}{a^4+2c^4}\ \forall\,\Delta\,ABC,\text{''}=\text{''}\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1728. In ΔABC holds:

$$\prod\sqrt[2024]{\frac{a}{b+c}}+\frac{R^{2024}}{r^{2024}}\geq\prod\sqrt[2024]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}+2^{2024}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\prod\sqrt[2024]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}\overset{AM-GM}{\leq}\sqrt[2024]{\frac{\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}{8\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}}=\frac{1}{2^{\frac{3}{2024}}}=\frac{1}{2^{\frac{3}{m}}},(m=2024),$$

$$\prod\sqrt[2024]{\frac{a}{b+c}}=\left(\frac{abc}{(a+b)(b+c)(c+a)}\right)^{\frac{1}{2024}}=$$

$$=\left(\frac{2Rr}{s^2+r^2+2Rr}\right)^{\frac{1}{2024}}\overset{Gerretsen}{\geq}\left(\frac{2Rr}{4R^2+6Rr+4r^2}\right)^{\frac{1}{2024}}\overset{Euler}{\geq}$$

$$\geq\left(\frac{1}{2}\frac{r^2}{R^2}\right)^{\frac{1}{2024}}\overset{Euler}{\geq}\left(\frac{r}{R}\right)^{\frac{3}{2024}}=\left(\frac{r}{R}\right)^{\frac{3}{m}}$$

$$\textit{Now we need to show }\left(\frac{r}{R}\right)^{\frac{3}{m}}+\left(\frac{R}{r}\right)^{m}\geq\frac{1}{2^{\frac{3}{m}}}+2^m\textit{ or}$$

$$(2x)^{\frac{3}{m}}(x^m-2^m)-\left(x^{\frac{3}{m}}-2^{\frac{3}{m}}\right)\overset{\frac{R}{r}=x\geq 2}{\geq}0$$

$$.\,let\ f(x)=(2x)^{\frac{3}{m}}(x^m-2^m)-\left(x^{\frac{3}{m}}-2^{\frac{3}{m}}\right)\ and$$

$$f'(x)=2^{\frac{3}{m}}\frac{3}{m}\,x^{\frac{3}{m}-1}(x^m-2^m)+(2x)^{\frac{3}{m}}\,mx^{m-1}-\frac{3}{m}x^{\frac{3}{m}-1}\ or,$$

$$f'(x)=x^{\frac{3}{m}-1}\left(2^{\frac{3}{m}}m\,x^m-\frac{3}{m}\right)+\frac{3}{m}x^{\frac{3}{m}-1}\left(2^{\frac{3}{m}}x^m-2^{\frac{3}{m}+m}\right)>0\ as\ x\geq 2.$$

$$f(2)=0\ so\ f(x)\geq f(2)=0\ hence\ (2x)^{\frac{3}{m}}(x^m-2^m)-\left(x^{\frac{3}{m}}-2^{\frac{3}{m}}\right)\geq 0\ (proved)$$

1729. In any acute triangle ABC, the following relationship holds :

$$a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}\ge 2p\sqrt[4]{3}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sin 2A+\sin 2B+\sin 2C=2\sin A\cos A+2\sin A\cos(B-C)$$
$$=2\sin A\left(\cos(B-C)-\cos(B+C)\right)=4\sin A\sin B\sin C=4.\frac{4Rrp}{8R^3}$$
$$\Rightarrow\sum_{cyc}\sin 2A=\frac{2rp}{R^2}\to(1)$$

$$\text{Now, } a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}=\sum_{cyc}\left((2R\sin A)\left(\sqrt{\frac{\sin A}{\cos A}}\right)\right)$$

$$=2R\sum_{cyc}\frac{\sin^2 A}{\sqrt{\sin A\cos A}}\overset{\text{Bergstrom}}{\ge}2R\frac{\left(\sum_{cyc}\sin A\right)^2}{\sum_{cyc}\sqrt{\sin A\cos A}}\overset{\text{CBS}}{\ge}\frac{2R\left(\frac{p}{R}\right)^2}{\sqrt{3}.\sqrt{\sum_{cyc}(\sin A\cos A)}}$$

$$=\frac{2\sqrt{2}R\left(\frac{p}{R}\right)^2}{\sqrt{3}.\sqrt{\sum_{cyc}\sin 2A}}\overset{\text{via }(1)}{=}\frac{2\sqrt{2}R\left(\frac{p}{R}\right)^2}{\sqrt{3}.\sqrt{\frac{2rp}{R^2}}}=\frac{2p.\sqrt{p}}{\sqrt{3}}\overset{\text{Mitrinovic}}{\ge}\frac{2p.\sqrt{3\sqrt{3}r}}{\sqrt{3r}}=2p\sqrt[4]{3}$$
$$\therefore a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}\ge 2p\sqrt[4]{3}$$
$$\forall\text{ acute }\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$let\ f(x)=\tan x\ ,x\in\left(0,\frac{\pi}{2}\right).f''(x)=2\sec^2 x\tan x>0,$$
$$f\text{ is convex}\in\left(0,\frac{\pi}{2}\right).\text{Using Jensen inequility}\sum\tan A\ge 3\tan\frac{\pi}{3}=3\sqrt{3}.$$
$$Note: A+B+C=\pi\text{ ,now }\tan(A+B)=\tan(\pi-C)\text{ or }\sum\tan A=\prod\tan A.$$
$$a\sqrt{\tan A}+b\sqrt{\tan B}+c\sqrt{\tan C}\overset{CHEBYSHEV}{\ge}$$
$$\ge\frac{1}{3}(a+b+c)\left(\sum\sqrt{tanA}\right)\overset{AM-GM}{\ge}\frac{1}{3}.2p.3\left(\prod tanA\right)^{\frac{1}{6}}\ge 2p.\left(3\sqrt{3}\right)^{\frac{1}{6}}=2p\sqrt[4]{3}$$

Equality holds for $a=b=c$.

1730. In ΔABC the following relationship holds:

$$min\left(\sum_{cyc}(b+c)r_a,\sum_{cyc}a(r_b+r_c)\right)\ge 36\sqrt{3}r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}(b+c)r_a=\sum_{cyc}(b+c)\cdot\frac{F}{s-a}=F\cdot\sum_{cyc}\frac{b+c}{s-a}=$$
$$=F\cdot\sum_{cyc}\frac{2s-a}{s-a}=F\cdot\sum_{cyc}\frac{s+s-a}{s-a}=F\left(s\sum_{cyc}\frac{1}{s-a}+3\right)=$$
$$=F\left(s\cdot\frac{4R+r}{rs}+3\right)=F\left(\frac{4R+r}{r}+3\right)\overset{EULER}{\geq}F\left(\frac{4\cdot 2r+r}{r}+3\right)=$$
$$=12F=12rs\overset{MITRINOVIC}{\geq}12r\cdot 3\sqrt{3}r=36\sqrt{3}r^2$$

$$\sum_{cyc}a(r_b+r_c)=\sum_{cyc}a\left(\frac{F}{s-b}+\frac{F}{s-c}\right)=F\sum_{cyc}a\cdot\frac{s-c+s-b}{(s-b)(s-c)}=$$
$$=F\sum_{cyc}\frac{a^2}{(s-b)(s-c)}=\frac{F}{(s-a)(s-b)(s-c)}\sum_{cyc}a^2(s-a)=$$
$$=\frac{Fs}{s(s-a)(s-b)(s-c)}\cdot 4rs(R+r)=\frac{Fs}{F^2}\cdot 4F(R+r)\overset{EULER}{\geq}4s(2r+r)=$$
$$=12rs\overset{MITRINOVIC}{\geq}12r\cdot 3\sqrt{3}r=36\sqrt{3}r^2$$

Equality holds for $a=b=c$.

1731. In ΔABC the following relationship holds:

$$\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{b}{c+a}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{c}{a+b}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\geq\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{b}{c+a}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{c}{a+b}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)=$$
$$=\sum_{cyc}\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}\frac{a}{b+c}\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=\sum_{cyc}\frac{a}{b+c}\cdot\frac{\sin\left(\frac{\pi-A}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=$$
$$=\sum_{cyc}\frac{a}{b+c}\cdot\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\prod_{cyc}\frac{a}{b+c}\cdot\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}}=$$

$$=3\cdot\sqrt[3]{\frac{abc}{(a+b)(b+c)(c+a)}\prod_{cyc}\frac{1}{cos\frac{A}{2}}}=3\cdot\sqrt[3]{\frac{4Rrs}{2s(s^2+r^2+2Rr)}\cdot\frac{4R}{s}}=$$

$$=6\cdot\sqrt[3]{\frac{R^2r}{s(s^2+r^2+2Rr)}}\overset{MITRINOVIC}{\geq}6\cdot\sqrt[3]{\frac{R^2r}{3\sqrt{3}r(s^2+r^2+2Rr)}}=$$

$$=\frac{6}{\sqrt{3}}\cdot\sqrt[3]{\frac{R^2}{s^2+r^2+2Rr}}\overset{GERRETSEN}{\geq}2\sqrt{3}\cdot\sqrt[3]{\frac{R^2}{4R^2+4Rr+3r^2+r^2+2Rr}}\overset{EULER}{\geq}$$

$$\geq 2\sqrt{3}\cdot\sqrt[3]{\frac{R^2}{4R^2+6R\cdot\frac{R}{2}+4\left(\frac{R}{2}\right)^2}}=2\sqrt{3}\cdot\sqrt[3]{\frac{R^2}{4R^2+6R\cdot\frac{R}{2}+4\left(\frac{R}{2}\right)^2}}=$$

$$=2\sqrt{3}\cdot\sqrt[3]{\frac{1}{8}}=\sqrt{3}$$

Equality holds for: $a=b=c$.

1732. In any Δ ABC, the following relationship holds :

$$\frac{h_a}{h_b+h_c}\left(tan\frac{B}{2}+tan\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}\left(tan\frac{C}{2}+tan\frac{A}{2}\right)+\frac{h_c}{h_a+h_b}\left(tan\frac{A}{2}+tan\frac{B}{2}\right)\geq\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C >0, (A + B), (B + C), (C + A) form sides of a triangle ($\because$ (A + B) + (B + C) $>$ (C + A) and analogs) $\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x} \overset{?}{\geq}$$

$$\frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{h_a}{h_b+h_c}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) + \frac{h_b}{h_c+h_a}\cdot\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)$$

$$+\frac{h_c}{h_a+h_b}\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(x = h_a, y = h_b, z = h_c, A = \tan\frac{A}{2}, B = \tan\frac{B}{2}, C = \tan\frac{C}{2}\right)$$

$$= \frac{x}{y+z}\cdot\sqrt{B+C}^2 + \frac{y}{z+x}\cdot\sqrt{C+A}^2 + \frac{z}{x+y}\cdot\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}\cdot\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc}\left(\tan\frac{A}{2}.\tan\frac{B}{2}\right)}$$

$$= \sqrt{3}.\sqrt{\frac{1}{s^2}\sum_{cyc} r_a r_b} = \sqrt{3}.\sqrt{\frac{1}{s^2}.s^2} = \sqrt{3}$$

$$\therefore \frac{h_a}{h_b+h_c}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) + \frac{h_b}{h_c+h_a}\cdot\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right) + \frac{h_c}{h_a+h_b}\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$$

$$\geq \sqrt{3}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1733.

In any Δ ABC, the following relationship holds :

$$\frac{a}{b+c}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) + \frac{b}{c+a}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right) + \frac{c}{a+b}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right) \geq 3\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle

$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc} AB + B^2\right) - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \to (1)$$

$$\text{Now}, \forall x,y,z>0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{a}{b+c}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)$$

$$= \frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a, y=b, z=c, A=\cot\frac{A}{2}, B=\cot\frac{B}{2}, C=\cot\frac{C}{2}\right)$$

$$= \frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}\cdot\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc}\left(\cot\frac{A}{2}.\cot\frac{B}{2}\right)}$$

$$= \sqrt{3}.\sqrt{s^2\sum_{cyc}\frac{1}{r_ar_b}} = \sqrt{3}.\sqrt{\frac{s^2(4R+r)}{rs^2}} \overset{\text{Euler}}{=\geq} \sqrt{3}.\sqrt{8+1} = 3\sqrt{3}$$

$$\therefore \frac{a}{b+c}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right) \geq 3\sqrt{3}$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1734.

In any Δ ABC, the following relationship holds :

$$\frac{a^2}{b^2+c^2}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{b^2}{c^2+a^2}\cdot\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{c^2}{a^2+b^2}\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) \geq \sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0, (A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$) \Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form
sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc} AB+B^2\right)-2\sum_{cyc} A^2-2\sum_{cyc} AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \to (1)$$

$$\text{Now}, \forall x,y,z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{a^2}{b^2+c^2}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) + \frac{b^2}{c^2+a^2}\cdot\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)$$

$$+\frac{c^2}{a^2+b^2}\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(x = a^2, y = b^2, z = c^2, A = \tan\frac{A}{2}, B = \tan\frac{B}{2}, C = \tan\frac{C}{2}\right)$$

$$= \frac{x}{y+z}\cdot\sqrt{B+C}^2 + \frac{y}{z+x}\cdot\sqrt{C+A}^2 + \frac{z}{x+y}\cdot\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}\cdot\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc}\left(\tan\frac{A}{2}.\tan\frac{B}{2}\right)}$$

$$= \sqrt{3}.\sqrt{\frac{1}{s^2}\sum_{cyc} r_a r_b} = \sqrt{3}.\sqrt{\frac{1}{s^2}.s^2} = \sqrt{3}$$

$$\therefore \frac{a^2}{b^2+c^2}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) + \frac{b^2}{c^2+a^2}\cdot\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right) + \frac{c^2}{a^2+b^2}\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$$

$$\geq \sqrt{3}\ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1735. In any Δ ABC, the following relationship holds :

$$\frac{a^n}{b^n+c^n}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) + \frac{b^n}{c^n+a^n}\cdot\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right) + \frac{c^n}{a^n+b^n}\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) \geq \sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle

$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and

$$16F^2 = 2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 =$$

$$= 2\sum_{cyc}\left(\sum_{cyc} AB + B^2\right) - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB =$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \to (1)$$

$$\text{Now}, \forall\, x,y,z > 0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} =$$

$$= \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow$$

$$\left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{a^n}{b^n+c^n}\cdot\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) + \frac{b^n}{c^n+a^n}\cdot\left(\tan\frac{C}{2} + \tan\frac{A}{2}\right)$$

$$+ \frac{c^n}{a^n+b^n}\cdot\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(x = a^n, y = b^n, z = c^n, A = \tan\frac{A}{2}, B = \tan\frac{B}{2}, C = \tan\frac{C}{2}\right)$$

$$= \frac{x}{y+z}\cdot\sqrt{B+C}^2 + \frac{y}{z+x}\cdot\sqrt{C+A}^2 + \frac{z}{x+y}\cdot\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F\cdot\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}\cdot\frac{\sqrt{3}}{2} = \sqrt{3}\cdot\sqrt{\sum_{cyc}\left(\tan\frac{A}{2}\cdot\tan\frac{B}{2}\right)}$$

$$= \sqrt{3}\cdot\sqrt{\frac{1}{s^2}\sum_{cyc} r_a r_b} = \sqrt{3}\cdot\sqrt{\frac{1}{s^2}\cdot s^2} = \sqrt{3}$$

$$\therefore \frac{a^n}{b^n+c^n}\cdot\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) + \frac{b^n}{c^n+a^n}\cdot\left(\tan\frac{C}{2} + \tan\frac{A}{2}\right) + \frac{c^n}{a^n+b^n}\cdot\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)$$

$$\geq \sqrt{3}\ \forall\, \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1736. In any Δ ABC, the following relationship holds:

$$\frac{a}{b+c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\ge 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, $(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\to(2)$$

$$\text{We have: }\frac{a}{b+c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a,y=b,z=c,A=\csc\frac{A}{2},B=\csc\frac{B}{2},C=\csc\frac{C}{2}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\ge}$$

$$4F\cdot\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\ge}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt3}{2}=\sqrt3\cdot\sqrt{\sum_{cyc}\left(\csc\frac{A}{2}\cdot\csc\frac{B}{2}\right)}$$

$$\overset{A-G}{\ge}\sqrt3\cdot\sqrt{3\cdot\sqrt[3]{\prod_{cyc}\csc^2\frac{A}{2}}}=\sqrt3\cdot\sqrt{3\cdot\sqrt[3]{\frac{16R^2}{r^2}}}\overset{\text{Euler}}{\ge}\sqrt3\cdot\sqrt{3\cdot\sqrt[3]{64}}=6$$

$$\therefore\frac{a}{b+c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\ge 6$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Mollweide's formula, we have

$$\frac{a}{b+c}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}\geq \sin\frac{A}{2}.\ \text{(and analogs)}$$

Then

$$\sum_{cyc}\frac{a}{b+c}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)\geq\sum_{cyc}\frac{\csc\frac{B}{2}+\csc\frac{C}{2}}{\csc\frac{A}{2}}=\sum_{cyc}\left(\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}}+\frac{\csc\frac{C}{2}}{\csc\frac{B}{2}}\right)\overset{AM-GM}{\geq}\sum_{cyc}2=6.$$

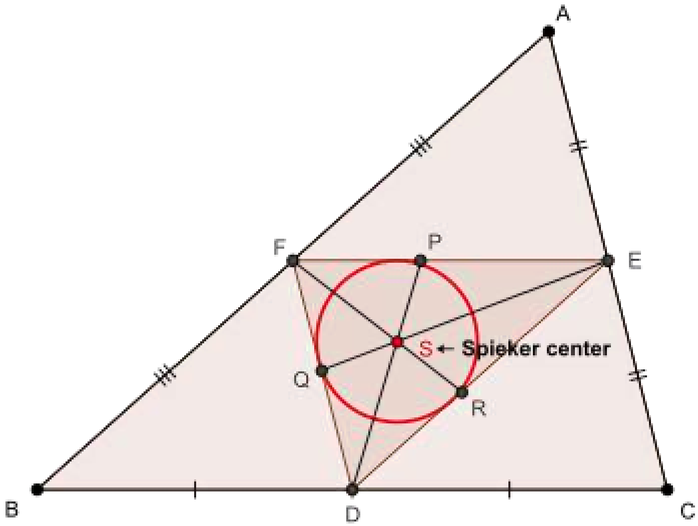
Equality holds iff ΔABC is equilateral.

1737. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{|b-c|}{w_a}\geq 2\sum_{cyc}\sqrt{\frac{p_a}{m_a}-1}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and m(∡BAX) = α and m(∡CAX) = β (say) and inradius of Δ DEF = r′(say)

$$\text{Now, }16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2}\ \textit{and}\ m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at : $AS^2=$

$$\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}$$
$$+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Again,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Also,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$

$$=rs\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}\big(b^3+c^3-abc\big)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)\big(b^3+c^3-abc\big)-\big(2b^2+2c^2-a^2\big)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\big(b^2-c^2\big)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\Big(\big(a^2+2a(b+c)+(b+c)^2\big)+\big((b+c)^2+2a(b+c)+a^2\big)-a^2\Big)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\big(2(a+b+c)^2-a^2\big)=\frac{(b-c)^2\big(8s^2-a^2\big)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2\big(8s^2-a^2\big)}{4(2s+a)^2}$$

$$\text{Now, }\frac{(b-c)^2}{w_a^2}\underset{(\blacksquare)}{\overset{?}{\geq}}4\left(\frac{p_a}{m_a}-1\right)=\frac{4\big(p_a^2-m_a^2\big)}{m_a(p_a+m_a)}\overset{\text{via }(\bullet)}{=}\frac{(b-c)^2\big(8s^2-a^2\big)}{(2s+a)^2m_a(p_a+m_a)}\text{ and }\because(b-c)^2$$

$$\geq 0\text{ and }m_a(p_a+m_a)\geq 2m_a^2\geq 2w_a^2$$

$$\left(\because p_a^2-m_a^2=\frac{(b-c)^2\big(8s^2-a^2\big)}{4(2s+a)^2}\overset{s>a}{\geq}0\therefore p_a\geq m_a\right)\therefore\text{ in order to prove }(\blacksquare),$$

$$\text{it suffices to prove}:1>\frac{8s^2-a^2}{2(2s+a)^2}\Leftrightarrow 8sa+3a^2>0\rightarrow\textit{true}\Rightarrow(\blacksquare)\text{ is true}$$

$$\therefore\frac{(b-c)^2}{w_a^2}\geq 4\left(\frac{p_a}{m_a}-1\right)\Rightarrow\frac{|b-c|}{w_a}\geq 2\sqrt{\frac{p_a}{m_a}-1}\text{ and analogs}$$

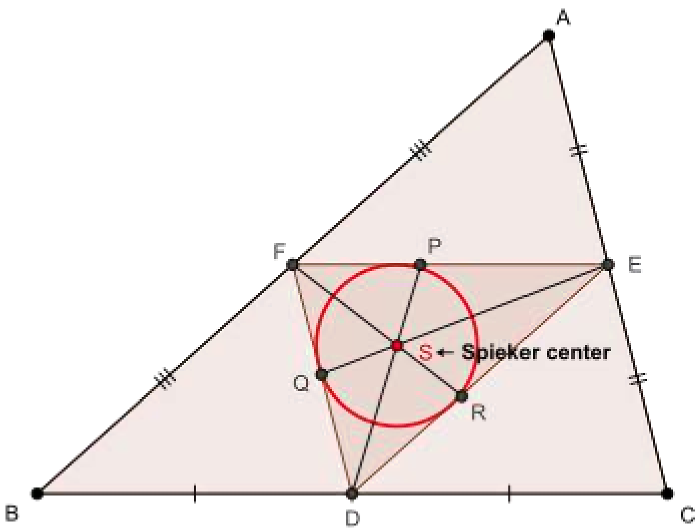
$$\Rightarrow\sum_{cyc}\frac{|b-c|}{w_a}\geq 2\sum_{cyc}\sqrt{\frac{p_a}{m_a}-1}\ \ \forall\,\Delta\,ABC,''=''\ \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1738. *In any $\Delta\, ABC$, the following relationship holds :*

$$m_a \geq max\left(\sqrt[3]{p_a r_b r_c}, \sqrt[3]{\frac{p_a h_a(r_b + r_c)}{2}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at : $AS^2 =$

$$\frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} = \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Again,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Also,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$

$$=rs \overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } m_a^3\overset{?}{\ge}p_a r_b r_c\Leftrightarrow m_a^6\overset{?}{\ge}p_a^2 s^2(s-a)^2\Leftrightarrow\frac{m_a^4}{s^2(s-a)^2}-1\overset{?}{\ge}\frac{p_a^2}{m_a^2}-1$$

$$\Leftrightarrow\frac{\left(m_a^2+s(s-a)\right)\left(m_a^2-s(s-a)\right)}{s^2(s-a)^2}\overset{?}{\ge}\frac{p_a^2-m_a^2}{m_a^2}$$

$$\overset{\text{via } (\bullet)}{\Leftrightarrow}\frac{\left(m_a^2+s(s-a)\right)\left(s(s-a)+\frac{(b-c)^2}{4}-s(s-a)\right)}{s^2(s-a)^2}\overset{?}{\ge}\frac{\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}}{m_a^2}$$

$$\Leftrightarrow\frac{\left(m_a^2+s(s-a)\right).\frac{(b-c)^2}{4}}{s^2(s-a)^2}\underset{(\blacksquare)}{\overset{?}{\ge}}\frac{\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}}{m_a^2}\text{ and }\because(b-c)^2\ge0\ \therefore\text{ in order}$$

$$\text{to prove } (\blacksquare)\text{, it suffices to prove : }\frac{m_a^2\left(m_a^2+s(s-a)\right)}{s^2(s-a)^2}\overset{(\blacksquare\blacksquare)}{>}\frac{8s^2-a^2}{(2s+a)^2}$$

$$\text{But, LHS of } (\blacksquare\blacksquare)\overset{\text{Lascu + A-G}}{\ge}\frac{s(s-a)\left(s(s-a)+s(s-a)\right)}{s^2(s-a)^2}=2\overset{?}{>}\frac{8s^2-a^2}{(2s+a)^2}$$

$$\Leftrightarrow 8s^2+8sa+2a^2\overset{?}{>}8s^2-a^2\Leftrightarrow 8sa+3a^2\overset{?}{>}0\to\text{true}\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true}$$

$$\therefore m_a^3\ge p_a r_b r_c\Rightarrow m_a\ge\sqrt[3]{p_a r_b r_c}\to(?)\ \therefore m_a^3\ge p_a r_b r_c\overset{?}{\ge}\frac{p_a h_a(r_b+r_c)}{2}$$

$$\Leftrightarrow 2s(s-a) \overset{?}{\geq} \frac{bc}{2R}.4R\cos^2\frac{A}{2} = 2bc.\frac{s(s-a)}{bc} \Leftrightarrow 2s(s-a) \overset{?}{\geq} 2s(s-a) \to \text{true}$$

$$\therefore m_a^3 \geq \frac{p_a h_a(r_b+r_c)}{2} \Rightarrow m_a \geq \sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}} \to (▯▯) \therefore (▯), (▯▯)$$

$$\Rightarrow m_a \geq \sqrt[3]{p_a r_b r_c}, \sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}}$$

$$\Rightarrow m_a \geq max\left(\sqrt[3]{p_a r_b r_c}, \sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}}\right), '' ='' \text{ iff } b = c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have the following known formulas (see [1, pp. 1]),

$$p_a{}^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}. \quad (1)$$

$$m_a{}^2 = s(s-a) + \frac{(b-c)^2}{4}, \quad (2)$$

By AM − GM inequality, we have

$$\sqrt[3]{p_a r_b r_c}^2 \leq \frac{p_a{}^2 + r_b r_c + r_b r_c}{3} \overset{(1)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{3(2s+a)^2}$$

$$= s(s-a) + \left(\frac{1}{4} - \frac{a(8s+3a)}{12(2s+a)^2}\right)(b-c)^2 \leq s(s-a) + \frac{(b-c)^2}{4} \overset{(2)}{=} m_a{}^2,$$

then

$$m_a \geq \sqrt[3]{p_a r_b r_c}.$$

Also, we have

$$\frac{h_a(r_b+r_c)}{2} = \frac{r_b r_c h_a}{2}\left(\frac{1}{r_b} + \frac{1}{r_c}\right) = r_b r_c.\frac{F}{a}.\frac{(s-b)+(s-c)}{F} = r_b r_c.$$

Therefore

$$m_a \geq \sqrt[3]{p_a r_b r_c} = max\left(\sqrt[3]{p_a r_b r_c}, \sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}}\right).$$

Equality holds if and only if $b = c$.

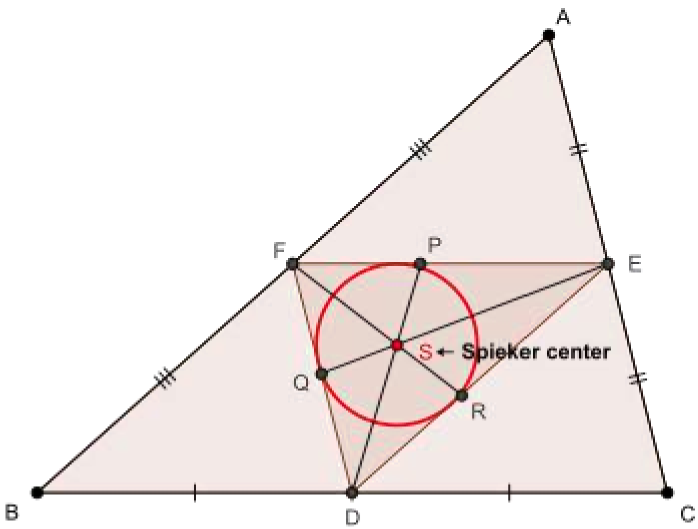
[1] **Bogdan Fuștei, Mohamed Amine Ben Ajiba,**
SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE-www.ssmrmh.ro

1739. **In any Δ ABC, the following relationship holds :**

$$R\sum_{cyc} h_a \geq \frac{4}{3}\sin\omega\sum_{cyc} p_a w_a$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at : $AS^2 =$

$$\frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} = \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Again,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Also,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$

$$= rs \overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right) - m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc) - \left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\bullet)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a^2w_a^2 \overset{?}{\leq} m_a^4 \Leftrightarrow \frac{p_a^2}{m_a^2} - 1 \overset{?}{\leq} \frac{m_a^2}{w_a^2} - 1 \overset{\text{via } (\bullet)}{\Leftrightarrow}$$

$$\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2m_a^2} \underset{(\blacksquare)}{\overset{?}{\leq}} \frac{s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)}{w_a^2}$$

$$= \frac{(b-c)^2(4s(s-a)+(b+c)^2)}{4(b+c)^2w_a^2} \text{ and } \because (b-c)^2 \geq 0 \text{ and } w_a^2 \leq m_a^2$$

$$\therefore \text{in order to prove } (\blacksquare), \text{it suffices to prove : } \frac{8s^2-a^2}{(2s+a)^2} < \frac{4s(s-a)+(b+c)^2}{(b+c)^2}$$

$$\Leftrightarrow \frac{8s^2-a^2}{(2s+a)^2} < \frac{4s(s-a)+(2s-a)^2}{(2s-a)^2} \Leftrightarrow \frac{8s^2-a^2}{(2s+a)^2} < \frac{8s^2-8sa+a^2}{(2s-a)^2}$$

$$\Leftrightarrow (8s^2-8sa+a^2)(2s+a)^2 > (8s^2-a^2)(2s-a)^2$$

$$\Leftrightarrow 16s^3-12s^2a-4sa^2+a^3 > 0 \Leftrightarrow 12s^2(s-a)+4s(s-a)(s+a)+a^3 > 0$$

$$\rightarrow \text{true} \because s-a > 0 \therefore p_a^2w_a^2 \leq m_a^4 \Rightarrow p_aw_a \leq m_a^2 \text{ and analogs}$$

$$\Rightarrow \frac{4}{3}\sin\omega . \sum_{cyc} p_aw_a \leq \frac{4}{3}.\frac{2rs}{\sqrt{\sum_{cyc} a^2b^2}}.\sum_{cyc} m_a^2$$

$$= \frac{4}{3}.\frac{2rs}{\sqrt{\sum_{cyc} a^2b^2}}.\frac{3}{4}.2(s^2-4Rr-r^2) \overset{?}{\leq} R.\sum_{cyc} h_a = \frac{\sum_{cyc} ab}{2}$$

$$\Leftrightarrow \frac{1}{4}\left(\sum_{cyc} ab\right)^2 \overset{?}{\geq} \frac{16r^2s^2(s^2-4Rr-r^2)^2}{\sum_{cyc} a^2b^2}$$

$$\Leftrightarrow \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} ab\right)^2 \underset{(?)}{\overset{?}{\geq}} 64r^2s^2(s^2-4Rr-r^2)^2$$

$$\text{Now, }\sum_{cyc} a^2b^2 \ge \frac{1}{3}\left(\sum_{cyc} ab\right)^2 \ge 8Rrs^2 \Rightarrow \text{LHS of } (*) \ge$$

$$8Rrs^2(s^2+4Rr+r^2)^2 \overset{?}{\ge} 64r^2s^2(s^2-4Rr-r^2)^2$$

$$\Leftrightarrow (R-2r)s^4 - 6rs^4 + rs^2(8R^2+66Rr+16r^2)$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) \underset{(**)}{\overset{?}{\ge}} 0$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\ge} \begin{pmatrix}(R-2r)(16Rr-5r^2)-6r(4R^2+4Rr+3r^2)\\ +r(8R^2+66Rr+16r^2)\end{pmatrix}s^2$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) = r^2(5R+8r)s^2$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) \overset{\text{Gerretsen}}{\ge} r^2(5R+8r)(16Rr-5r^2)$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) \overset{?}{\ge} 0 \Leftrightarrow 2t^3-5t^2+5t-6 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2t^2-t+3) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true} \Rightarrow$$

$$R.\sum_{cyc} h_a \ge \frac{4}{3}\sin\omega.\sum_{cyc} p_a w_a \ \forall \Delta ABC,'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1740. In ΔABC the following relationship holds:

$$a^a \cdot b^b \cdot c^c \cdot (m_a+m_b+m_c)^{2s} \ge (2s)^{2s} \cdot m_a^2 \cdot m_b^b \cdot m_c^c$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

$$a^a \cdot b^b \cdot c^c \cdot (m_a+m_b+m_c)^{2s} \ge (2s)^{2s} \cdot m_a^a \cdot m_b^b \cdot m_c^c$$

$$\left(\frac{a}{m_a}\right)^a \cdot \left(\frac{b}{m_b}\right)^b \cdot \left(\frac{c}{m_c}\right)^c \ge \left\{\frac{(a+b+c)}{m_a+m_b+m_c}\right\}^{2s}$$

$$\left(\frac{a}{m_a}\right)^a \cdot \left(\frac{b}{m_b}\right)^b \cdot \left(\frac{c}{m_c}\right)^c \ge \left(\frac{a+b+c}{m_a+m_b+m_c}\right)^{a+b+c}$$

GM ≥ HM

$$\left[\left(\frac{a}{m_a}\right)^a \cdot \left(\frac{b}{m_b}\right)^b \cdot \left(\frac{c}{m_c}\right)^c\right]^{\frac{1}{a+b+c}} \ge \frac{a+b+c}{\frac{a}{\left(\frac{a}{m_a}\right)}+\frac{b}{\left(\frac{b}{m_b}\right)}+\frac{c}{\left(\frac{c}{m_c}\right)}} = \frac{a+b+c}{m_a+m_b+m_c}$$

$$\therefore \left(\frac{a}{m_a}\right)^a \cdot \left(\frac{b}{m_b}\right)^b \cdot \left(\frac{c}{m_c}\right)^c \ge \left(\frac{a+b+c}{m_a+m_b+m_c}\right)^{a+b+c}$$

Equality holds for $a=b=c$.

1741. In ΔABC the following relationship holds:

$$a\sqrt{4a^2+9b^2}+b\sqrt{4b^2+9c^2}+c\sqrt{4c^2+9a^2}\geq 10\sqrt{6}\cdot F$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

$$4a^2+9b^2\geq\frac{(2a+3b)^2}{2}$$

$$a\sqrt{4a^2+9b^2}+b\sqrt{4b^2+9c^2}+c\sqrt{4c^2+9a^2}\geq\sum\frac{a(2a+3b)}{\sqrt{2}}=$$

$$=\sqrt{2}\sum a^2+\frac{3}{\sqrt{2}}\sum ab\geq\sqrt{2}\sum ab+\frac{3}{\sqrt{2}}\sum ab\overset{Gordon}{\geq}$$

$$\geq\sqrt{2}\cdot 4\sqrt{3}F+\frac{3}{\sqrt{2}}\cdot 4\sqrt{3}F=4\sqrt{6}F+6\sqrt{6}F=10\sqrt{6}F$$

Equality holds for $a=b=c$.

1742.

In any Δ ABC, the following relationship holds :

$$2R\sum_{cyc}\cos\frac{A-B}{2}\leq\sum_{cyc}\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2$$

Proposed by Eldeniz Hesenov-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2=\frac{m_a^2}{\frac{2bc}{b^2+c^2}.m_a\cos^2\frac{A}{2}}\overset{A-G}{\geq}\frac{m_a}{\cos^2\frac{A}{2}}\overset{Lascu}{\geq}\frac{\frac{b+c}{2}.\cos\frac{A}{2}}{\cos^2\frac{A}{2}}$$

$$=\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2\cos\frac{A}{2}}\Rightarrow\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2\geq 2R\cos\frac{B-C}{2}\text{ and analogs}\Rightarrow$$

$$\sum_{cyc}\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2 \ge 2R\sum_{cyc}\cos\frac{A-B}{2}\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$4m_a^2 = 2b^2+2c^2-a^2 = 2bc\cos A + b^2+c^2, Now\ 4m_a^2 \overset{AM-GM}{\ge} 2bc(1+\cos A)\ or,$$

$$m_a \ge \sqrt{bc}\cos\frac{A}{2}\ (1),$$

$$\left(\frac{m_a}{\left(\sqrt{s_a}\cos\frac{A}{2}\right)}\right)^2 = \frac{m_a^2(b^2+c^2)}{2bcm_a\cos^2\frac{A}{2}} =$$

$$= \frac{m_a(b^2+c^2)}{2\,bc\cos^2\frac{A}{2}} \overset{(1)\&CBS}{\ge} \frac{\sqrt{bc}\cos\frac{A}{2}(b+c)^2}{4\,bc\cos^2\frac{A}{2}} \overset{AM-GM}{\ge}$$

$$\ge \frac{b+c}{2}\frac{1}{\cos\frac{A}{2}} = \frac{R(\sin B+\sin C)}{\cos\frac{A}{2}} = \frac{2R\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{A}{2}} = 2R\cos\frac{B-C}{2}$$

using this result we get

$$2R\sum\cos\left(\frac{A-B}{2}\right) \le \sum\left(\frac{m_a}{\left(\sqrt{s_a}\cos\frac{A}{2}\right)}\right)^2$$

1743. **In ΔABC the following relationship holds:**

$$\frac{bch_a{r_a}^2}{2bc+h_a}+\frac{ach_b{r_b}^2}{2ac+h_b}+\frac{abh_c{r_c}^2}{2ab+h_c} \ge \frac{324r^4}{4R+1}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{bch_a{r_a}^2}{2bc+h_a}+\frac{ach_b{r_b}^2}{2ac+h_b}+\frac{abh_c{r_c}^2}{2ab+h_c} = \frac{{r_a}^2}{\frac{2}{h_a}+\frac{1}{bc}}+\frac{{r_b}^2}{\frac{2}{h_b}+\frac{1}{ac}}+\frac{{r_c}^2}{\frac{2}{h_c}+\frac{1}{ab}} \ge$$

$$\ge \frac{(r_a+r_b+r_c)^2}{2\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)+\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)} = \frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{a+b+c}{abc}} = \frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{2P}{abc}} =$$

$$=\frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{2S}{r\cdot 4RS}}=\frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{1}{2Rr}}=\frac{(r_a+r_b+r_c)^2}{\frac{4R+1}{2Rr}}=\frac{2Rr(r_a+r_b+r_c)^2}{4R+1}\overset{Euler}{\geq}$$

$$\geq\frac{4r^2(r_a+r_b+r_c)^2}{4R+1}\quad(*)$$

$$\begin{cases}r_a+r_b=4Rcos^2\frac{C}{2}\\ r_b+r_c=4Rcos^2\frac{A}{2}\\ r_a+r_c=4Rcos^2\frac{B}{2}\end{cases}\Rightarrow r_a+r_b+r_c=2R\left(cos^2\frac{A}{2}+cos^2\frac{B}{2}+cos^2\frac{C}{2}\right)=$$

$$=R(1+cosA+1+cosB+1+cosC)=R(3+cosA+cosB+cosC)=$$

$$=R\left(3+\left(1+\frac{r}{R}\right)\right)=R\cdot\frac{4R+r}{R}=4R+r\quad(1)$$

$$(*)\overset{(1)}{\Rightarrow}\frac{4r^2(4R+r)^2}{4R+1}\overset{Euler}{\geq}\frac{4r^2(8r+r)^2}{4R+1}=\frac{324r^4}{4R+1}$$

Equality holds for $a=b=c$.

1744. In *all* non − isosceles Δ ABC, the following identity is true :

$$\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{r(3s^2-r^2-4Rr)}{4R^3}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$b-c=4R\sin\frac{A}{2}\sin\frac{B-C}{2}\Rightarrow\sin\frac{B-C}{2}=\frac{b-c}{4R\sin\frac{A}{2}}\text{ and analogs}$$

$$\therefore\frac{4R^3}{r}\cdot\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{4R^3}{r}\cdot\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\frac{a-b}{4R\sin\frac{C}{2}}\cdot\frac{a-c}{4R\sin\frac{B}{2}}}$$

$$=\frac{4R^3.16R^2}{r}.\sum_{cyc}\frac{\sin^4 A\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}{(a-b)(a-c)}=$$

$$=\frac{4R^3.16R^2}{r}.\sum_{cyc}\frac{\sin^4 A\left(\frac{r}{4R}\right)}{(a-b)(a-c)}=\sum_{cyc}\frac{(2R\sin A)^4}{(a-b)(a-c)}=$$

$$=\frac{a^4(b-c)}{(a-b)(a-c)(b-c)}+\frac{b^4(c-a)}{(b-a)(b-c)(c-a)}+\frac{c^4(a-b)}{(c-a)(c-b)(a-b)}$$

$$=\frac{a^4(b-c)+b^4(c-a)+c^4(a-b)}{(a-b)(a-c)(b-c)}=\frac{a^4(b-c)+(b^4c-bc^4)-a(b^4-c^4)}{(a-b)(a-c)(b-c)}$$

$$=\frac{(b-c)\left((a^4-ab^3)+(b^3c-ab^2c)+(b^2c^2-abc^2)-(ac^3-bc^3)\right)}{(a-b)(a-c)(b-c)}$$

$$=\frac{(b-c)(a-b)\left((a^3-c^3)+b(a^2-c^2)+b^2(a-c)\right)}{(a-b)(a-c)(b-c)}$$

$$=\frac{(b-c)(a-b)(a-c)(a^2+c^2+ca+ab+bc+b^2)}{(a-b)(a-c)(b-c)}=\sum_{cyc}a^2+\sum_{cyc}ab$$

$$=2(s^2-4Rr-r^2)+s^2+4Rr+r^2=3s^2-r^2-4Rr$$

$$\therefore\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{r(3s^2-r^2-4Rr)}{4R^3}\ \text{(QED)}$$

1745. In any Δ ABC, the following relationship holds :

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq 2s$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Rovsen Pirguliyev-Azerbaijan

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq 2s \quad (*)$$

To prove that :

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq a+b+c \quad (1)$$

$$\left(\frac{a^2+bc}{b+c}-a\right)+\left(\frac{b^2+ca}{c+a}-b\right)+\left(\frac{c^2+ab}{a+b}-c\right)\geq 0$$

$$\frac{a^2+bc-ab-ca}{b+c}+\frac{b^2+ca-bc-ab}{c+a}+\frac{c^2+ab-ac-bc}{a+b}\geq 0$$

or $(a-b)\cdot\left(\frac{a-c}{b+c}-\frac{b-c}{c+a}\right)+(b-c)\cdot\left(\frac{b-a}{c+a}-\frac{c-a}{a+b}\right)+(c-a)\cdot\left(\frac{c-b}{a+b}-\frac{a-b}{b+c}\right)\geq 0$ or

$$(a-b)^2\cdot\frac{a+b}{(b+c)(c+a)}+(b-c)^2\cdot\frac{b+c}{(c+a)(a+b)}+\frac{(c-a)^2\cdot(c+a)}{(a+b)(b+c)}\geq 0$$

$a+b+c=2s$ then true (*)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^2b+\sum_{cyc}ab^2=\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)=\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc$$

$$=\sum_{cyc}a^2b+\sum_{cyc}ab^2\rightarrow(1)$$

Now, $\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}=\sum_{cyc}\frac{a^4}{a^2b+a^2c}+\sum_{cyc}\frac{b^2c^2}{b^2c+bc^2}\overset{\text{Bergstrom}}{\geq}$

$$\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b+\sum_{cyc}ab^2}+\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}a^2b+\sum_{cyc}ab^2}\overset{A-G}{\geq}\frac{2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)}{\sum_{cyc}a^2b+\sum_{cyc}ab^2}$$

$$\overset{\text{via }(1)}{=}\frac{2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)}{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc}=\frac{4(s^2-4Rr-r^2)(s^2+4Rr+r^2)}{2s(s^2+4Rr+r^2)-12Rrs}$$

$$=\frac{2(s^4-(4Rr+r^2)^2)}{s(s^2-2Rr+r^2)}\overset{?}{\geq}2s\Leftrightarrow s^4-(4Rr+r^2)^2\overset{?}{\geq}s^4-s^2(2Rr-r^2)$$

$$\Leftrightarrow(2R-r)s^2\underbrace{\overset{?}{\geq}}_{(*)}r(4R+r)^2$$

$$\text{Now, }(2R-r)s^2\overset{\text{Gerretsen}}{\geq}(2R-r)(16Rr-5r^2)\overset{?}{\geq}r(4R+r)^2$$

$\Leftrightarrow 8R^2-17Rr+2r^2\overset{?}{\geq}0\Leftrightarrow(8R-r)(R-2r)\overset{?}{\geq}0\rightarrow$ true via Euler $\Rightarrow(*)$ is true

$\therefore\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq 2s\ \forall\ \Delta ABC,''=''$ iff ΔABC is equilateral (QED)

Solution 3 by Șerban George Florin-Romania

$$\sum_{cyc}\frac{a^2+bc}{b+c}\geq 2s,\sum_{cyc}\left(\frac{a^2+bc}{b+c}+a\right)\geq 2s+\sum_{cyc}a,$$

$$\sum_{cyc}\frac{a^2+bc+ab+ac}{b+c}\ge 2\sum_{cyc}a,$$

$$\sum_{cyc}\frac{(a+b)(a+c)}{b+c}\ge 2\sum_{cyc}a, a+b=z, b+c=x, a+c=y,$$

$$\sum_{cyc}\frac{yz}{x}\ge\sum_{cyc}x, \sum_{cyc}x=p, \sum_{cyc}xy=q, r=\prod_{cyc}x, \sum_{cyc}\frac{yz}{x}=\frac{1}{xyz}\sum_{cyc}(yz)^2\ge\sum_{cyc}x,$$

$\frac{q^2-2pr}{r}\ge p, q^2-2pr\ge pr, q^2\ge 3pr$, true, Schur inequality. Then : $\sum_{cyc}\frac{a^2+bc}{b+c}\ge 2s$

equality is if $a=b=c$.

1746. **If $x\in\mathbb{R}$ then in any ΔABC the following relationship holds:**

$$a\sum\sqrt{(a\sin x)^2+(b\cos x)^2}\ge 2\sqrt{6}\cdot F$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\sum a\sqrt{a^2\sin^2 x+b^2\cos^2 x}\ge\sum a\frac{a|\sin x|+b|\cos x|}{\sqrt{2}}=$$

$$=\frac{1}{\sqrt{2}}\Big((a^2+b^2+c^2)|\sin x|+(ab+ac+bc)|\cos x|\Big)\ge$$

$$\ge\frac{1}{\sqrt{2}}(ab+ac+bc)(|\sin x|+|\cos x|)\ge\frac{1}{\sqrt{2}}\cdot 4\sqrt{3}F\cdot 1=\frac{4\sqrt{6}F}{2}=2\sqrt{6}F$$

$$f(x)=\sin x+\cos x\quad x\in\left(0;\frac{\pi}{2}\right)$$

$$f'(x)=\cos x-\sin x=0\Rightarrow x=\frac{\pi}{4}$$

0	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$f'(x)$	+ + + + + + + + + + + + +	0 − − − − − − − − − − − − − − − − − −	
$f(x)$	1 ↗	$\sqrt{2}$ ↘	1

$$\Rightarrow f(x)\ge 1$$

Solution 2 by Tapas Das-India

$$\sum a\sqrt{(a\sin x)^2+(b\cos x)^2}=\sum\sqrt{(a^2\sin x)^2+(ab\cos x)^2}\ge$$

$$\ge\sqrt{\left(|\sin x|\sum a^2\right)^2+\left(|\cos x|\sum ab\right)^2}\ (Minkowski)\ge$$

$$\ge\frac{1}{\sqrt{2}}\left[|\sin x|\sum a^2+|\cos x|\sum ab\right]\ge$$

$$\ge\frac{1}{\sqrt{2}}\left[|\sin x|\sum ab+|\cos x|\sum ab\right]=$$

$$=\frac{1}{\sqrt{2}}\sum ab\,(|\sin x|+|\cos x|)\overset{Gordon}{\ge}4\sqrt{3}\frac{F}{\sqrt{2}}=2\sqrt{6}\,F$$

$$Note: (|\sin x|+|\cos x|)^2=1+|\sin 2x|\ge 1+0=1.$$

$$|\sin x|+|\cos x|\ge 1$$

1747. In any Δ ABC, the following relationship holds :

$$9\sum_{cyc}\tan^3\frac{A}{2}\tan\frac{B}{2}+\frac{(4R+r)^2}{p^2}\ge 6$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$9\sum_{cyc}\tan^3\frac{A}{2}\tan\frac{B}{2}=9\sum_{cyc}\frac{\tan^3\frac{A}{2}\tan^3\frac{B}{2}}{\tan^2\frac{B}{2}}\overset{Radon}{\ge}9.\frac{\left(\sum_{cyc}\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)\right)^3}{\left(\sum_{cyc}\tan\frac{A}{2}\right)^2}$$

$$=9.\frac{\left(\frac{1}{p^2}\sum_{cyc}r_ar_b\right)^3}{\left(\frac{1}{p}\sum_{cyc}r_a\right)^2}=9.\frac{\left(\frac{1}{p^2}.p^2\right)^3}{\left(\frac{4R+r}{p}\right)^2}=\frac{9p^2}{(4R+r)^2}\Rightarrow$$

$$9\sum_{cyc}\tan^3\frac{A}{2}\tan\frac{B}{2}+\frac{(4R+r)^2}{p^2}\ge\frac{9p^2}{(4R+r)^2}+\frac{(4R+r)^2}{p^2}\overset{A-G}{\ge}$$

$$\ge 2.\sqrt{\frac{9p^2}{(4R+r)^2}\cdot\frac{(4R+r)^2}{p^2}}=6$$

$$\therefore 9\sum_{cyc} tan^3\frac{A}{2}tan\frac{B}{2}+\frac{(4R+r)^2}{p^2}\geq 6\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1748. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{m_a m_b}{m_a+m_b-m_c}\geq\sum_{cyc}m_a$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall prove that } \forall\ \Delta\ ABC: \sum_{cyc}\frac{ab}{a+b-c}\geq\sum_{cyc}a\to(1) \text{ and } (1)$$

$$\Leftrightarrow\sum_{cyc}\frac{ab}{2(s-c)}\geq 2s\Leftrightarrow\sum_{cyc}\frac{bc}{s(s-a)}\geq 4\Leftrightarrow\sum_{cyc}\sec^2\frac{A}{2}\geq 4\Leftrightarrow\frac{(4R+r)^2+s^2}{s^2}\geq 4$$

$\Leftrightarrow (4R+r)^2\geq 3s^2\to$ true via Trucht (Doucet) $\therefore$ (1) is true and implementing (1) on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$, we arrive at :

$$\sum_{cyc}\frac{\frac{4}{9}.m_a m_b}{\frac{2}{3}(m_a+m_b-m_c)}\geq\frac{2}{3}\sum_{cyc}m_a\ \therefore\sum_{cyc}\frac{m_a m_b}{m_a+m_b-m_c}\geq\sum_{cyc}m_a$$

$$\forall\ \Delta\ ABC,''='' \text{ iif } \Delta\ ABC \text{ is equilateral (QED)}$$

1749. In ΔABC the following relationship holds:

$$r_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)+r_b\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)+r_c\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)\leq\frac{9R}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\sin\frac{A}{2}\overset{Jensen}{\leq}3\sin\left(\frac{A+B+C}{6}\right)=\frac{3}{2},$$

$$WLOG\ a\geq b\geq c\ then\ r_a\geq r_b\geq r_c\ and\sin\frac{A}{2}\geq\sin\frac{B}{2}\geq\sin\frac{C}{2}$$

$$r_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)+r_b\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)+r_c\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)\overset{Chebyshev}{\leq}$$

$$\leq\frac{1}{3}\left(\sum r_a\right)\left(\sum\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\right)\leq\frac{1}{3}(4R+r).2.\frac{3}{2}\overset{Euler}{\leq}\frac{9R}{2}$$

Equality holds for $a=b=c$.

1750.

In any Δ ABC, the following relationship holds :

$$\frac{h_a}{h_b+h_c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{h_c}{h_a+h_b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C >0, (A + B), (B + C), (C + A) form sides of a triangle ($\because$ (A + B) + (B + C) > (C + A) and analogs) $\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

Now, $\forall x,y,z>0$, $\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$

Via Bergstrom, LHS of $(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

We have : $\frac{h_a}{h_b+h_c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)$

$$+\frac{h_c}{h_a+h_b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$\left(x=h_a,y=h_b,z=h_c,A=\csc\frac{A}{2},B=\csc\frac{B}{2},C=\csc\frac{C}{2}\right)$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{via\ (1)\ and\ (2)}{\geq} 2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}} = \sqrt{3}.\sqrt{\sum_{cyc}\left(\csc\frac{A}{2}.\csc\frac{B}{2}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3}.\sqrt{3.\sqrt[3]{\prod_{cyc}\csc^2\frac{A}{2}}} = \sqrt{3}.\sqrt{3.\sqrt[3]{\frac{16R^2}{r^2}}} \overset{Euler}{\geq} \sqrt{3}.\sqrt{3.\sqrt[3]{64}} = 6$$

$$\therefore \frac{h_a}{h_b+h_c}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{h_c}{h_a+h_b}.\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)$$

$$\geq 6\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1751. In any Δ ABC, the following relationship holds :

$$\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}+\csc\frac{C}{2}}.(b+c)+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}+\csc\frac{A}{2}}.(c+a)+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}+\csc\frac{B}{2}}.(a+b)\geq 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\ A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and

$$16F^2 = 2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 =$$

$$2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now}, \forall\ x, y, z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

Via Bergstrom,

$$\text{LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} =$$

$$\frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true}$$

$$\therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{\csc\frac{A}{2}}{\csc\frac{B}{2}+\csc\frac{C}{2}}.(b+c) + \frac{\csc\frac{B}{2}}{\csc\frac{C}{2}+\csc\frac{A}{2}}.(c+a) + \frac{\csc\frac{C}{2}}{\csc\frac{A}{2}+\csc\frac{B}{2}}.(a+b)$$

$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(x = \csc\frac{A}{2}, y = \csc\frac{B}{2}, z = \csc\frac{C}{2}, A = a, B = b, C = c\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}.\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc} ab}$$

$$\overset{\text{Gordon}}{\geq} \sqrt{3}.\sqrt{4\sqrt{3}rs} \overset{\text{Mitrinovic}}{\geq} 6.\sqrt{3\sqrt{3}r^2} = 6\sqrt{3}r$$

$$\therefore \frac{\csc\frac{A}{2}}{\csc\frac{B}{2}+\csc\frac{C}{2}}.(b+c) + \frac{\csc\frac{B}{2}}{\csc\frac{C}{2}+\csc\frac{A}{2}}.(c+a) + \frac{\csc\frac{C}{2}}{\csc\frac{A}{2}+\csc\frac{B}{2}}.(a+b) \geq 6\sqrt{3}r$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1752. In any ΔABC and $\forall n \in \mathbb{N}$, the following relationships hold :

$$① \frac{a^n}{b^n+c^n}.\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) + \frac{b^n}{c^n+a^n}.\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right) + \frac{c^n}{a^n+b^n}.\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right) \geq 3\sqrt{3}$$

$$② \frac{h_a^n}{h_b^n+h_c^n}.\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) + \frac{h_b^n}{h_c^n+h_a^n}.\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right) + \frac{h_c^n}{h_a^n+h_b^n}.\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right) \geq 3\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall A, B, C > 0$, $(A+B), (B+C), (C+A)$ form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{a^n}{b^n+c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)$$

$$+\frac{c^n}{a^n+b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a^n,y=b^n,z=c^n,A=\cot\frac{A}{2},B=\cot\frac{B}{2},C=\cot\frac{C}{2}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\cot\frac{A}{2}\cdot\cot\frac{B}{2}\right)}$$

$$=\sqrt{3}.\sqrt{s^2\sum_{cyc}\frac{1}{r_ar_b}}=\sqrt{3}.\sqrt{s^2\cdot\frac{4R+r}{rs^2}}\overset{\text{Euler}}{\geq}\sqrt{3}.\sqrt{\frac{8r+r}{r}}=3\sqrt{3}$$

$$\therefore\frac{a^n}{b^n+c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{c^n}{a^n+b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)$$

$$\geq 3\sqrt{3}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

$$\text{Again, we have : }\frac{h_a^n}{h_b^n+h_c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)$$

$$+\frac{h_c^n}{h_a^n+h_b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=h_a^n,y=h_b^n,z=h_c^n,A=\cot\frac{A}{2},B=\cot\frac{B}{2},C=\cot\frac{C}{2}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\cot\frac{A}{2}\cdot\cot\frac{B}{2}\right)}$$

$$=\sqrt{3}.\sqrt{s^2\sum_{cyc}\frac{1}{r_a r_b}}=\sqrt{3}.\sqrt{s^2.\frac{4R+r}{rs^2}}\overset{Euler}{\geq}\sqrt{3}.\sqrt{\frac{8r+r}{r}}=3\sqrt{3}$$

$$\therefore\frac{h_a^n}{h_b^n+h_c^n}.\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}.\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}.\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\geq 3\sqrt{3}\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

Walter Janous inequality : for a, b, c and x, y, z be positive real numbers :

$$\frac{x}{y+z}(b+c)+\frac{y}{z+x}(c+a)+\frac{z}{x+y}(a+b)\geq\sqrt{3(ab+bc+ca)}$$

Using this solution of (1)&(2)

$$1)\ \frac{a^n}{b^n+c^n}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{c^n}{a^n+b^n}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\geq$$

$$\geq\sqrt{3\sum\cot\frac{A}{2}\cot\frac{B}{2}}=\sqrt{\frac{3s^2(r_a+r_b+r_c)}{r_ar_br_c}}=\sqrt{\frac{3(4R+r)}{r}}\overset{Euler}{\geq}\left(\frac{27r}{r}\right)^{\frac{1}{2}}=3\sqrt{3}$$

$$2)\ \frac{h_a^n}{h_b^n+h_c^n}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\geq$$

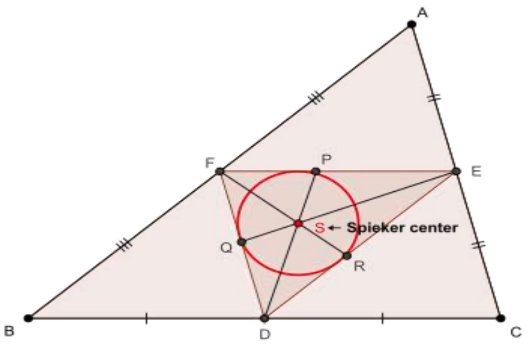
$$\geq\sqrt{3\sum\cot\frac{A}{2}\cot\frac{B}{2}}=\sqrt{\frac{3s^2(r_a+r_b+r_c)}{r_ar_br_c}}=\sqrt{\frac{3(4R+r)}{r}}\overset{Euler}{\geq}\left(\frac{27r}{r}\right)^{\frac{1}{2}}=3\sqrt{3}$$

1753. **In any Δ ABC with $p_a, p_b, p_c \to$ Spieker cevians, the following relationship holds :**

$$\frac{p_a-m_a+w_a}{h_a}+\frac{p_b-m_b+w_b}{h_b}+\frac{p_c-m_c+w_c}{h_c}\leq\frac{R}{r}+1$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again}, \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2 s}\big(ca(s-b) + ab(s-c)\big) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\big(b^2+c^2+ab+ca\big) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now}, [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{Also}, p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big) - m_a^2$$

$$= \frac{2s}{(2s+a)^2}\big(b^3+c^3-abc\big) - \left(1 - \frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)\big(b^3+c^3-abc\big) - \big(2b^2+2c^2-a^2\big)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2\big(b^2-c^2\big)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\Big(\big(a^2+2a(b+c)+(b+c)^2\big) + \big((b+c)^2+2a(b+c)+a^2\big) - a^2\Big)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\big(2(a+b+c)^2 - a^2\big) = \frac{(b-c)^2\big(8s^2-a^2\big)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\blacksquare\blacksquare)}{=} \frac{(b-c)^2\big(8s^2-a^2\big)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \geq w_a \Rightarrow w_a \leq p_a$$

$$\therefore \text{ in order to prove : } \frac{p_a^2-m_a^2}{p_a+m_a}\le\frac{m_a^2-w_a^2}{m_a+w_a}, \text{ it suffices to prove : }$$
$$p_a^2-m_a^2\le m_a^2-w_a^2$$
$$\overset{\text{via }(\blacksquare\blacksquare)}{\Leftrightarrow}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\le s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$
$$=\frac{(b-c)^2}{4}\left(1+\frac{4s(s-a)}{(2s-a)^2}\right)=\frac{(b-c)^2}{4}\cdot\frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$
$$\Leftrightarrow\left((2s-a)^2+4s(s-a)\right)(2s+a)^2\ge(8s^2-a^2)(2s-a)^2$$
$$\Leftrightarrow 16s^3-12s^2a-4sa^2+a^3\ge 0\Leftrightarrow(s-a)(16s^2+4sa)+a^3\ge 0$$
$$\to\text{true (strict) since } s>a\ \therefore\frac{p_a^2-m_a^2}{p_a+m_a}\le\frac{m_a^2-w_a^2}{m_a+w_a}\Rightarrow p_a+w_a\le 2m_a \text{ and analogs}$$
$$\Rightarrow\sum_{cyc}\frac{p_a-m_a+w_a}{h_a}\le\sum_{cyc}\frac{m_a}{h_a}=\frac{1}{2rs}\sum_{cyc}(\sqrt{a}m_a.\sqrt{a})\overset{CBS}{\le}$$
$$\frac{1}{4rs}.\sqrt{\sum_{cyc}a(2b^2+2c^2-a^2)}.\sqrt{\sum_{cyc}a}$$
$$=\frac{\sqrt{2s}}{4rs}.\sqrt{2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-6abc-2s(s^2-6Rr-3r^2)}$$
$$=\frac{\sqrt{2s}.\sqrt{2s}}{4rs}.\sqrt{2(s^2+4Rr+r^2)-12Rr-s^2+6Rr+3r^2}\overset{Gerretsen}{\le}$$
$$\frac{1}{2r}.\sqrt{4R^2+4Rr+3r^2+2Rr+5r^2}=\frac{1}{2r}.\sqrt{4R^2+6Rr+4r^2+4r^2}$$
$$\overset{Euler}{\le}\frac{1}{2r}.\sqrt{4R^2+6Rr+4r^2+2Rr}=\frac{1}{2r}.\sqrt{4R^2+8Rr+4r^2}=\frac{1}{r}.\sqrt{(R+r)^2}$$
$$=\frac{R}{r}+1\ \therefore\frac{p_a-m_a+w_a}{h_a}+\frac{p_b-m_b+w_b}{h_b}+\frac{p_c-m_c+w_c}{h_c}\le\frac{R}{r}+1$$
$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1754.

In any Δ ABC with p_a, p_b, p_c $\to$ Spieker's cevians, the following relationship holds

$$: p_a+p_b+p_c\le m_a+m_b+m_c+\frac{1}{3}(max\{a,b,c\}-min\{a,b,c\})$$

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Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at : $AS^2 =$

$$\frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} = \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Again, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Also, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES},\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$

$$=rs\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)\left(b^3+c^3-abc\right)-\left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right)=\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}$$

Now, we shall prove that :

$$\frac{1}{2}\left(|b-c|+|c-a|+|a-b|\right)\overset{(\blacklozenge)}{=}\max\{a,b,c\}-\min\{a,b,c\}$$

Case (1) $a\ge b\ge c\therefore\frac{1}{2}\left(|b-c|+|c-a|+|a-b|\right)=\frac{1}{2}(b-c+a-c+a-b)$
$=a-c=\max\{a,b,c\}-\min\{a,b,c\}$

Case (2) $a\ge c\ge b\therefore\frac{1}{2}\left(|b-c|+|c-a|+|a-b|\right)=\frac{1}{2}(c-b+a-c+a-b)$
$=a-b=\max\{a,b,c\}-\min\{a,b,c\}$

Case (3) $b\ge c\ge a\therefore\frac{1}{2}\left(|b-c|+|c-a|+|a-b|\right)=\frac{1}{2}(b-c+c-a+b-a)$
$=b-a=\max\{a,b,c\}-\min\{a,b,c\}$

Case (4) $b\ge a\ge c\therefore\frac{1}{2}\left(|b-c|+|c-a|+|a-b|\right)=\frac{1}{2}(b-c+a-c+b-a)$
$=b-c=\max\{a,b,c\}-\min\{a,b,c\}$

$$\boxed{Case\ (5)}\ c \ge a \ge b \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(c-b+c-a+a-b)$$
$$= c-b = max\{a,b,c\} - \min\{a,b,c\}$$
$$\boxed{Case\ (6)}\ c \ge b \ge a \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(c-b+c-a+b-a)$$
$$= c-a = max\{a,b,c\} - \min\{a,b,c\}$$
$$\therefore \text{combining all 6 cases, we conclude}: \frac{1}{2}(|b-c|+|c-a|+|a-b|)$$
$$= max\{a,b,c\} - \min\{a,b,c\}$$
$$\therefore \text{via } (\blacklozenge), p_a + p_b + p_c \le m_a + m_b + m_c + \frac{1}{3}(max\{a,b,c\} - \min\{a,b,c\})$$
$$\Leftrightarrow \sum_{cyc} p_a \overset{(\blacksquare)}{\le} \sum_{cyc} m_a + \frac{1}{6}\sum_{cyc}|b-c|$$
$$\text{Now}, a^2(4m_a^2) = a^2(2b^2+2c^2-a^2)$$
$$= 2a^2b^2 + 2c^2a^2 + 2b^2c^2 - a^4 - b^4 - c^4 + (b^4+c^4-2b^2c^2)$$
$$= 16F^2 + (b^2-c^2)^2 > (b^2-c^2)^2 \Rightarrow 2am_a > |b^2-c^2|$$
$$\Rightarrow m_a > \frac{|b-c|(b+c)}{2a} \therefore m_a > \frac{|b-c|(2s-a)}{2a} \Rightarrow \left(m_a + \frac{|b-c|}{6}\right)^2$$
$$= m_a^2 + \frac{(b-c)^2}{36} + m_a.\frac{|b-c|}{3} \ge m_a^2 + \frac{(b-c)^2}{36} + \frac{|b-c|(2s-a)}{2a}.\frac{|b-c|}{3} \overset{?}{\ge} p_a^2$$
$$\Leftrightarrow \frac{(2s-a)(b-c)^2}{6a} \overset{?}{\ge} p_a^2 - m_a^2 - \frac{(b-c)^2}{36} \overset{\text{via }(\bullet)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} - \frac{(b-c)^2}{36}$$
$$= \frac{9(8s^2-a^2)-(2s+a)^2}{36(2s+a)^2}.(b-c)^2 = \frac{68s^2-4sa-10a^2}{36(2s+a)^2}.(b-c)^2$$
$$\Leftrightarrow 3(2s-a)(2s+a)^2 \overset{?}{\ge} a(34s^2-2sa-5a^2)\ (\because (b-c)^2 \ge 0)$$
$$\Leftrightarrow 12s^3 - 11s^2a - 2sa^2 + a^3 \overset{?}{\ge} 0 \Leftrightarrow (s-a)\left(12s^2 + a(s-a)\right) \overset{?}{\ge} 0 \to \text{true}$$
$$(\text{strict}) \because s > a \therefore m_a + \frac{|b-c|}{6} \ge p_a \text{ and analogs}$$
$$\Rightarrow \sum_{cyc} m_a + \frac{1}{6}\sum_{cyc}|b-c| \ge \sum_{cyc} p_a \Rightarrow (\blacksquare) \text{ is true}$$
$$\therefore p_a + p_b + p_c \le m_a + m_b + m_c + \frac{1}{3}(max\{a,b,c\} - \min\{a,b,c\})$$
$$\forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

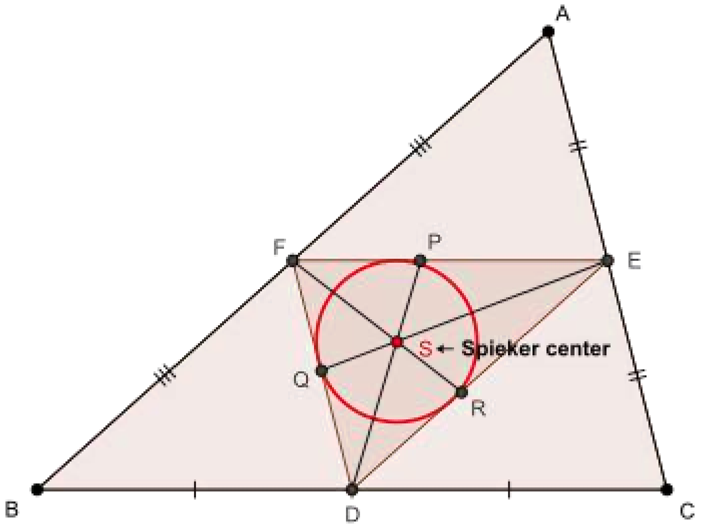
1755.

In any Δ ABC, the following relationship holds :

$$p_a + p_b + p_c \ge m_a + m_b + m_c + \frac{a^2+b^2+c^2-ab-bc-ca}{3s}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC at X and } m(\measuredangle BAX) = \alpha \text{ and } m(\measuredangle CAX) = \beta \text{ (say)}$$
$$\text{and inradius of } \Delta\, DEF = r' \text{(say)}$$
$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$
$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$
$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$
$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$
$$\text{Via (1), (2) and using cosine law on } \Delta\, AFS \text{ and } \Delta\, AES, \text{ we arrive at : } AS^2 =$$
$$\frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} = \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}}$$
$$+ \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\text{Again, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Also,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS,}\frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on }\Delta\text{ AES, } b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$

$$= rs \overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2-m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$

$$= \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)\left(b^3+c^3-abc\right)-\left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a\overset{?}{\geq}m_a+\frac{(b-c)^2}{6s}\Leftrightarrow p_a^2\overset{?}{\geq}m_a^2+\frac{(b-c)^4}{36s^2}+\frac{(b-c)^2}{3s}.m_a$$

$$\Leftrightarrow p_a^2-m_a^2\overset{?}{\geq}\frac{(b-c)^4}{36s^2}+\frac{(b-c)^2}{3s}.m_a\overset{\text{via }(\bullet)}{\Leftrightarrow}$$

$$\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\underset{(\blacklozenge)}{\overset{?}{\geq}}\frac{(b-c)^4}{36s^2}+\frac{(b-c)^2}{3s}.m_a$$

$$\text{Since } m_a<\frac{b+c}{2}=\frac{2s-a}{2}\text{ and }(b-c)^2<a^2\ \therefore\text{ in order to prove }(\blacklozenge),$$

$$\text{it suffices to prove : }\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}-\frac{(b-c)^2a^2}{36s^2}\geq\frac{(b-c)^2}{3s}.\frac{2s-a}{2}$$

$$\Leftrightarrow\frac{9s^2(8s^2-a^2)-a^2(2s+a)^2}{36s^2(2s+a)^2}\geq\frac{2s-a}{6s}\ \left(\because (b-c)^2\geq 0\right)$$

$$\Leftrightarrow 24s^4-24s^3a-s^2a^2+2sa^3-a^4\geq 0\Leftrightarrow (s-a)\left(23s^3+s(s-a)(s+a)+a^3\right)$$

$$\geq 0\to\text{true (strict)}\because s>a\Rightarrow(\blacklozenge)\text{ is true}$$

$$\therefore p_a\overset{?}{\geq}m_a+\frac{(b-c)^2}{6s}\text{ and analogs}\Rightarrow\sum_{cyc}p_a\overset{?}{\geq}\sum_{cyc}m_a+\frac{1}{6s}.\sum_{cyc}(b-c)^2$$

$$\therefore p_a+p_b+p_c\geq m_a+m_b+m_c+\frac{a^2+b^2+c^2-ab-bc-ca}{3s}$$

$$\forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1756.

In any $\Delta\,ABC$ with p_a, p_b, p_c
$\to$ Spieker cevians, the following relationship holds :
$$h_a(p_a+w_a)+h_b(p_b+w_b)+h_c(p_c+w_c)\leq 2s^2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of $\Delta\,DEF=r'$ (say)

$$\text{Now, }16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

Via sine law on Δ AFS, $\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$

$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}$ and via sine law on Δ AES, $b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$

Now, $[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

Also, $p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$

$$=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)\left(b^3+c^3-abc\right)-\left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right)=\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\blacksquare\blacksquare)}{=}\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}\overset{s>a}{\geq}0\Rightarrow p_a\geq m_a\geq w_a\Rightarrow w_a\leq p_a$$

$\therefore$ in order to prove : $\frac{p_a^2-m_a^2}{p_a+m_a}\leq\frac{m_a^2-w_a^2}{m_a+w_a}$, it suffices to prove :

$$p_a^2-m_a^2\leq m_a^2-w_a^2$$

$$\overset{\text{via }(\blacksquare\blacksquare)}{\Leftrightarrow}\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}\leq s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$=\frac{(b-c)^2}{4}\left(1+\frac{4s(s-a)}{(2s-a)^2}\right)=\frac{(b-c)^2}{4}\cdot\frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$

$$\Leftrightarrow\left((2s-a)^2+4s(s-a)\right)(2s+a)^2\geq\left(8s^2-a^2\right)(2s-a)^2$$

$$\Leftrightarrow 16s^3 - 12s^2a - 4sa^2 + a^3 \ge 0 \Leftrightarrow (s-a)(16s^2+4sa) + a^3 \ge 0$$

$$\to \text{true (strict) since } s > a \therefore \frac{p_a^2 - m_a^2}{p_a + m_a} \le \frac{m_a^2 - w_a^2}{m_a + w_a} \Rightarrow p_a + w_a \le 2m_a \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} h_a(p_a + w_a) \le 4rs \sum_{cyc} \frac{m_a}{a} \overset{?}{\le} 2s^2 \Leftrightarrow \frac{s}{2r} \overset{?}{\ge} \sum_{cyc} \frac{m_a}{a} \Leftrightarrow \frac{s}{2r} \overset{?}{\ge} \frac{1}{4Rrs} \sum_{cyc} bcm_a$$

$$\Leftrightarrow (2Rs^2)^2 \overset{?}{\ge} \left(\sum_{cyc} bcm_a\right)^2$$

$$= \sum_{cyc} \left(b^2c^2\left(\frac{2b^2+2c^2-a^2}{4}\right)\right) + 2\sum_{cyc}(bc.ca.m_am_b)$$

$$\Leftrightarrow 16R^2s^4 \overset{?}{\ge} \sum_{cyc}\left(b^2c^2\left(2\sum_{cyc}a^2 - 3a^2\right)\right) + 32Rrs\sum_{cyc} cm_am_b$$

$$\Leftrightarrow 16R^2s^4 \underset{(\bullet)}{\overset{?}{\ge}} 4(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2 - 16Rrs^2\right)$$

$$-144R^2r^2s^2 + 32Rrs\sum_{cyc} cm_am_b$$

$$\text{Now, } m_am_b \overset{?}{\le} \frac{2c^2+ab}{4} \Leftrightarrow \left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right) \overset{?}{\le} \frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4 + b^4 - 2a^2b^2 - a^2c^2 + 2abc^2 - b^2c^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a+b)^2(a-b)^2 - c^2(a-b)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a-b)^2(a+b+c)(a+b-c) \overset{?}{\ge} 0 \to \text{true} \Rightarrow m_am_b \le \frac{2c^2+ab}{4} \text{ and analogs}$$

$$\therefore \text{RHS of } (\bullet) \le 4(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2 - 16Rrs^2\right) - 144R^2r^2s^2$$

$$+32Rrs\sum_{cyc}\left(c.\frac{2c^2+ab}{4}\right) \overset{?}{\le} 16R^2s^4$$

$$\Leftrightarrow (s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2 - 16Rrs^2\right) - 36R^2r^2s^2$$

$$+8Rrs(4(s^2-6Rr-3r^2)+12Rrs) \overset{?}{\le} 4R^2s^4$$

$$\Leftrightarrow s^6 - (4R^2+4Rr-r^2)s^4 - (12R^2+16Rr+r^2)r^2s^2 - r^3(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\le}} 0$$

$$\text{Now, Rouche} \Rightarrow s^2 - (m-n) \ge 0 \text{ and } s^2 - (m+n) \le 0, \text{where } m =$$
$$2R^2 + 10Rr - r^2 \text{ and } n = 2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \le 0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3 \le 0$$

$$\Rightarrow s^6 - (4R^2+20Rr-2r^2)s^4 + r(4R+r)^3.s^2 \le 0 \therefore \text{in order to prove } (\bullet\bullet),$$

it suffices to prove :

$$s^6 - (4R^2+4Rr-r^2)s^4 - (12R^2+16Rr+r^2)r^2s^2 - r^3(4R+r)^3$$
$$\le s^6 - (4R^2+20Rr-2r^2)s^4 + r(4R+r)^3.s^2$$

$$\Leftrightarrow (16R-5r)s^4-(64R^3+60R^2r+28Rr^2+2r^3)s^2-r^2(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\leq} 0$$

$$\text{Again, LHS of } (\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\leq}$$

$$\left((16R-5r)(4R^2+4Rr+3r^2)-(64R^3+60R^2r+28Rr^2+2r^3)\right)s^2$$

$$-r^2(4R+r)^3 \overset{?}{\leq} 0 \Leftrightarrow (16R-5r)s^2 \underbrace{\overset{?}{\leq}}_{(\bullet\bullet\bullet\bullet)} (4R+r)^3$$

$$\text{Moreover, LHS of } (\bullet\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\leq} (16R-5r)(4R^2+4Rr+3r^2) \overset{?}{\leq} (4R+r)^3$$

$$\Leftrightarrow 4r(R-2r)^2 \overset{?}{\geq} 0 \to \text{true} \Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore h_a(p_a+w_a)+h_b(p_b+w_b)+h_c(p_c+w_c) \leq 2s^2$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1757. In acute ΔABC the following relationship holds:

$$(b+c)secA+(c+a)secB+(a+b)secC \geq 24\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$(b+c)secA+(c+a)secB+(a+b)secC \underbrace{\geq}^{AM-GM}$$

$$\geq 3\cdot\sqrt[3]{\prod_{cyc}(a+b)\cdot\prod_{cyc}\frac{1}{cosA}} \underbrace{\geq}^{CESARO} 3\cdot\sqrt[3]{8abc\cdot\prod_{cyc}\frac{1}{cosA}} \geq$$

$$\geq 6\cdot\sqrt[3]{abc\cdot\frac{1}{\frac{1}{8}}} = 12\cdot\sqrt[3]{abc} = 12\cdot\sqrt[3]{4Rrs} \underbrace{\geq}^{EULER} 12\cdot\sqrt[3]{8r^2s} \geq$$

$$\underbrace{\geq}^{MITRINOVIC} 24\cdot\sqrt[3]{r^2\cdot 3\sqrt{3}r} = 24r\cdot\sqrt[3]{(\sqrt{3})^3} = 24\sqrt{3}r$$

Equality holds for $a=b=c$.

1758.

In any acute Δ ABC, the following relationship holds :

$$\frac{a}{b+c}.(\sec B+\sec C)+\frac{b}{c+a}.(\sec C+\sec A)+\frac{c}{a+b}.(\sec A+\sec B) \geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB + B^2\right) - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB + 2\sum_{cyc}A^2 - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \to (1)$$

$$\text{Now, } \forall\, x, y, z > 0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy + z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2 + xyz\sum_{cyc}x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have: } \frac{a}{b+c}.(\sec B + \sec C) + \frac{b}{c+a}.(\sec C + \sec A) + \frac{c}{a+b}.(\sec A + \sec B)$$

$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$(x = a, y = b, z = c, A = \sec A, B = \sec B, C = \sec C)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc}\sec A\sec B}$$

$$= \sqrt{3\left(\prod_{cyc}\sec A\right)\left(\sum_{cyc}\cos A\right)} = \sqrt{\frac{12R^2(R+r)}{R(s^2-4R^2-4Rr-r^2)}} \overset{\text{Gerretsen}}{\geq}$$

$$\sqrt{\frac{12R(R+r)}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}} = \sqrt{\frac{6R(R+r)}{r^2}} \overset{\text{Euler}}{\geq} \sqrt{\frac{12r(3r)}{r^2}} = 6$$

$$\therefore \frac{a}{b+c}.(\sec B + \sec C) + \frac{b}{c+a}.(\sec C + \sec A) + \frac{c}{a+b}.(\sec A + \sec B) \geq 6$$

$\forall$ $acute$ Δ ABC, " = " iff Δ ABC is equilateral (QED)

1759.

In any acute Δ ABC, the following relationship holds:

$$\frac{\sec A}{\sec B + \sec C}.(b+c) + \frac{\sec B}{\sec C + \sec A}.(c+a) + \frac{\sec C}{\sec A + \sec B}.(a+b) \geq 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle
($\because$ (A + B) + (B + C) > (C + A) and analogs) $\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now}, \forall x,y,z>0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{\sec A}{\sec B+\sec C}.(b+c)+\frac{\sec B}{\sec C+\sec A}.(c+a)+\frac{\sec C}{\sec A+\sec B}.(a+b)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$(x=\sec A, y=\sec B, z=\sec C, A=a, B=b, C=c)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}ab}$$

$$\overset{\text{Gordon}}{\geq}\sqrt{3}.\sqrt{4\sqrt{3}rs}\overset{\text{Mitrinovic}}{\geq}\sqrt{3}.\sqrt{4\sqrt{3}.3\sqrt{3}r^2}=6\sqrt{3}r$$

$$\therefore\frac{\sec A}{\sec B+\sec C}.(b+c)+\frac{\sec B}{\sec C+\sec A}.(c+a)+\frac{\sec C}{\sec A+\sec B}.(a+b)\geq 6\sqrt{3}r$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

1760.

In any Δ ABC, the following relationship holds :

$$① \frac{a^n}{b^n+c^n}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c^n}{a^n+b^n}.\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

$$② \frac{h_a^n}{h_b^n+h_c^n}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}.\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia
Solution by Soumava Chakraborty-Kolkata-India

$$\forall A,B,C>0,(A+B),(B+C),(C+A) \text{ form sides of a triangle}$$

$(\because (A+B)+(B+C)>(C+A)$ *and analogs*$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with *area* F (*say*) *and*

$$16F^2=2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=$$

$$=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB=$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{x}{y+z}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{y}{z+x}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{z}{x+y}.\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)\ \left(A=\csc\frac{A}{2},B=\csc\frac{B}{2},C=\csc\frac{C}{2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\csc\frac{A}{2}\csc\frac{B}{2}}$$

$$\overset{A-G}{\geq}3.\sqrt[6]{\csc^2\frac{A}{2}\csc^2\frac{B}{2}\csc^2\frac{C}{2}}=3.\sqrt[6]{\frac{16R^2}{r^2}}\overset{\text{Euler}}{\geq}3.\sqrt[6]{\frac{64r^2}{r^2}}=6$$

$$\therefore\forall x,y,z>0,\frac{x}{y+z}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{y}{z+x}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)$$

$$+\frac{z}{x+y}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

$\therefore$ choosing $x=a^n, y=b^n, z=c^n$ *and*
$x=h_a^n, y=h_b^n, z=h_c^n$ separately, we arrive at :

$$\frac{a^n}{b^n+c^n}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c^n}{a^n+b^n}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

and

$$\frac{h_a^n}{h_b^n+h_c^n}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

respectively, $\forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

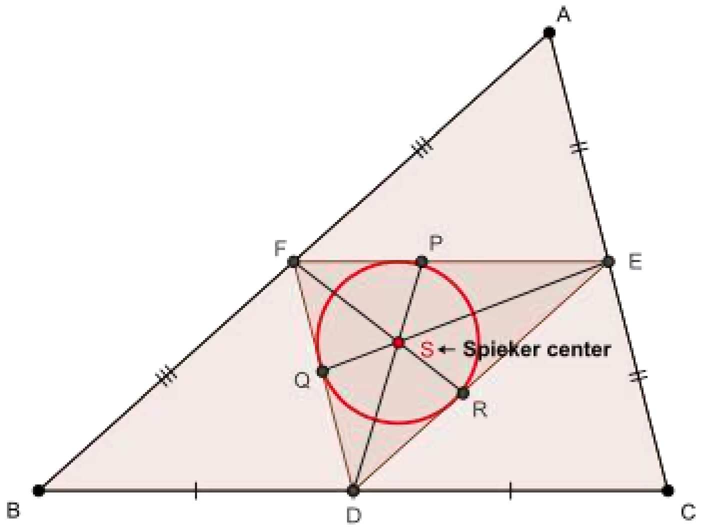
1761.

In any Δ ABC with p_a, p_b, p_c
$\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c}\geq 3$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say)
and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2}\ and\ m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on } \Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\ \text{and via sine law on } \Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{We have}:\prod_{\text{cyc}}(2s+a)=8s^3+4s^2\sum_{\text{cyc}}a+2s\sum_{\text{cyc}}ab+4Rrs$$

$$=8s^3+4s^2.2s+2s\big(s^2+4Rr+r^2\big)+4Rrs$$

$$\Rightarrow\prod_{\text{cyc}}(2s+a)\overset{(\blacksquare\blacksquare)}{=}2s\big(9s^2+6Rr+r^2\big)$$

$$\text{Now, } b^3+c^3-abc+a\big(4m_a^2\big)=b^3+c^3+a^3-abc+a\big(2b^2+2c^2-a^2\big)-a^3$$

$$=\sum_{\text{cyc}}a^3-abc+2a.2bc\cos A=2s\big(s^2-6Rr-3r^2\big)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a\big(4m_a^2\big)\overset{(\blacksquare\blacksquare\blacksquare)}{=}2s\big(s^2-8Rr-3r^2+8Rr\cos A\big)\ \therefore(\blacksquare),(\blacksquare\blacksquare\blacksquare)$$

$$\Rightarrow p_a=\frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}$$

$$=\frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\left(1-2\sin^2\frac{A}{2}\right)}$$

$$\Rightarrow p_a=\frac{2s}{2s+a}.\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}\ \text{and analogs}$$

$$\Rightarrow\frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c}=\sum_{\text{cyc}}\frac{(2s+a)r_a}{2s.\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}}$$

$$=\frac{\prod_{cyc}(2s+a)}{2s}\cdot\sum_{cyc}\frac{r_a}{\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}(2s+b)(2s+c)}$$

$$\overset{\text{via }(\blacksquare\blacksquare)}{=}\frac{2s(9s^2+6Rr+r^2)}{2s}\cdot$$

$$\sum_{cyc}\frac{r_a^2}{\sqrt{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(2s+b)(2s+c)r_a}\cdot\sqrt{(2s+b)(2s+c)r_a}}$$

$$\overset{\text{Bergstrom}}{\geq}$$

$$\binom{9s^2+6Rr}{+r^2}\cdot\frac{\left(\sum_{cyc}r_a\right)^2}{\sum_{cyc}\left(\sqrt{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(2s+b)(2s+c)r_a}\cdot\sqrt{(2s+b)(2s+c)r_a}\right)}$$

$$\overset{\text{CBS}}{\geq}\frac{(9s^2+6Rr+r^2)(4R+r)^2}{\sqrt{\sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(2s+b)(2s+c)r_a\right)}\cdot\sqrt{\sum_{cyc}((2s+b)(2s+c)r_a)}}$$

$$\Rightarrow\frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c}\overset{(\blacklozenge)}{\geq}$$

$$\frac{(9s^2+6Rr+r^2)(4R+r)^2}{\sqrt{\sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(8s^2-2sa+bc)r_a\right)}\cdot\sqrt{\sum_{cyc}((8s^2-2sa+bc)r_a)}}$$

We have : $\sum_{cyc}\frac{r_a}{a}=\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}=\frac{s}{4R}\sum_{cyc}\sec^2\frac{A}{2}=\frac{s}{4R}\cdot\frac{s^2+(4R+r)^2}{s^2}$

$$\Rightarrow\sum_{cyc}\frac{r_a}{a}\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}\frac{s^2+(4R+r)^2}{4Rs}\text{ and }\sum_{cyc}ar_a=rs\sum_{cyc}\frac{a-s+s}{s-a}$$

$$=rs\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)\Rightarrow\sum_{cyc}ar_a\overset{(\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare)}{=}(4R-2r)s$$

We now proceed to evaluate : $\sum_{cyc}\left(-16Rr\sin^2\frac{A}{2}(8s^2-2sa+bc)r_a\right)$

Firstly, $\sum_{cyc}\left(-8s^2.16Rr\sin^2\frac{A}{2}r_a\right)=-128Rrs^2\sum_{cyc}\left(\left(1-\cos^2\frac{A}{2}\right)s\tan\frac{A}{2}\right)$

$$=-128Rrs^2(4R+r)+32rs^3\sum_{cyc}\left(4R\cos\frac{A}{2}\sin\frac{A}{2}\right)$$

$$=-128Rrs^2(4R+r)+32rs^3(2s)$$

$$\therefore\sum_{cyc}\left(-8s^2.16Rr\sin^2\frac{A}{2}r_a\right)=-128Rrs^2(4R+r)+64rs^4\rightarrow(3)$$

$$\text{Secondly, } \sum_{cyc}\left(\left(-16Rr\sin^2\frac{A}{2}\right)(-2sa)\right) = 32Rrs\sum_{cyc}\left(\left(1-\cos^2\frac{A}{2}\right)ar_a\right)$$

$$= 32Rrs\left(\sum_{cyc} ar_a - \sum_{cyc}\left(a.s\tan\frac{A}{2}.\cos^2\frac{A}{2}\right)\right) \overset{via\ (■■■■■)}{=}$$

$$32Rrs\left((4R-2r)s - \frac{sa}{4R}.\left(4R\cos\frac{A}{2}\sin\frac{A}{2}\right)\right)$$

$$= 32Rrs^2(4R-2r) - 8rs^2\sum_{cyc} a^2 = 32Rrs^2(4R-2r) - 16rs^2(s^2-4Rr-r^2)$$

$$\therefore \sum_{cyc}\left(\left(-16Rr\sin^2\frac{A}{2}\right)(-2sa)\right) = 32Rrs^2(4R-2r) - 16rs^2(s^2-4Rr-r^2) \to (4)$$

$$\text{Thirdly, } \sum_{cyc}\left(-16Rr\sin^2\frac{A}{2}bcr_a\right) = -16Rr.4Rrs.\sum_{cyc}\frac{\sin^2\frac{A}{2}.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}.\tan\frac{A}{2}}$$

$$= -16Rr^2\sum_{cyc} r_a^2 \therefore \sum_{cyc}\left(-16Rr\sin^2\frac{A}{2}bcr_a\right) = -16Rr^2\left((4R+r)^2-2s^2\right) \to (5)$$

$$\text{Again, } \sum_{cyc}\left((8s^2-2sa+bc)r_a\right) = 8s^2(4R+r) - 2s\sum_{cyc} ar_a + 4Rrs.\sum_{cyc}\frac{r_a}{a}$$

$$\overset{via\ (■■■■)\ and\ (■■■■■)}{=} 8s^2(4R+r) - 2s^2(4R-2r) + 4Rrs.\frac{s^2+(4R+r)^2}{4Rs}$$

$$\therefore \sum_{cyc}\left((8s^2-2sa+bc)r_a\right) = (24R+13r)s^2 + r(4R+r)^2 \to (6)$$

$$\text{Via } (3),(4),(5),(6), \sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(8s^2-2sa+bc)r_a\right)$$

$$= (s^2-3r^2)\left((24R+13r)s^2+r(4R+r)^2\right) - 128Rrs^2(4R+r) + 64rs^4$$
$$+32Rrs^2(4R-2r) - 16rs^2(s^2-4Rr-r^2) - 16Rr^2\left((4R+r)^2-2s^2\right)$$

$$\therefore \sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(8s^2-2sa+bc)r_a\right)$$
$$= (24R+61r)s^4 - r(368R^2+160Rr+22r^2)s^2$$
$$-r^2(256R^3+176R^2r+40Rr^2+3r^3) \to (7)$$

$$\therefore (6),(7) \text{ and } (◆) \Rightarrow \frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c} \geq$$

$$\frac{(9s^2+6Rr+r^2)(4R+r)^2}{\sqrt{(24R+61r)s^4 - r(368R^2+160Rr+22r^2)s^2 - r^2(256R^3+176R^2r+40Rr^2+3r^3)}}.$$

$$\frac{1}{\sqrt{(24R+13r)s^2+r(4R+r)^2}} \overset{?}{\geq} 3$$

$$\Leftrightarrow (9s^2+6Rr+r^2)^2(4R+r)^4 \overset{?}{\geq}$$

$$9\begin{pmatrix}(24R+61r)s^4 - r(368R^2+160Rr+22r^2)s^2 \\ -r^2(256R^3+176R^2r+40Rr^2+3r^3)\end{pmatrix}\left((24R+13r)s^2+r(4R+r)^2\right)$$

$$\Leftrightarrow -(5184R^2+15984Rr+7137r^2)s^6$$
$$+(20736R^4+96768R^3r+74880R^2r^2+20160Rr^3+2106r^4)s^4$$
$$+r(27648R^5+140544R^4r+132480R^3r^2+50688R^2r^3+8748Rr^4+567r^5)s^2$$
$$+r^2\begin{pmatrix}9216R^6+49152R^5r+50560R^4r^2+22720R^3r^3+5220R^2r^4+604Rr^5\\+28r^6\end{pmatrix}\boxed{\overset{?}{\underset{(\bullet)}{\geq}}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$
$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$

$\therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet)\geq$

$$-(5184R^2+15984Rr+7137r^2)(s^2)\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)$$
$$-4r(17712R^3+65745R^2r+22653Rr^2-4095r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}$$
$$\Leftrightarrow (9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4-3132r^5)s^2$$
$$+r\begin{pmatrix}567936R^6+2535072R^5r+2415368R^4r^2+818798R^3r^3+71163R^2r^4\\-13168Rr^5-2044r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet)}{\geq}}0$$

$\boxed{Case\ 1}$ $9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4-3132r^5\geq 0$ *and* then : LHS of $(\bullet\bullet)\geq$

$$\begin{pmatrix}9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4\\-3132r^5\end{pmatrix}(16Rr-5r^2)$$
$$+r\begin{pmatrix}567936R^6+2535072R^5r+2415368R^4r^2+818798R^3r^3+71163R^2r^4\\-13168Rr^5-2044r^6\end{pmatrix}\overset{?}{\geq}0$$
$$\Leftrightarrow 360000t^6+187248t^5-2523815t^4+1048165t^3+851814t^2-225140t+6808\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}360000t^4+1627248t^3+2545177t^2\\+4719881t+9550630\end{pmatrix}+19097856\right)\overset{?}{\geq}0$$
$$\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet)\text{ is true}$$

$\boxed{Case\ 2}$ $9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4-3132r^5<0$ *and* then : LHS of $(\bullet\bullet)\geq$

$$\begin{pmatrix}9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3\\+77400Rr^4-3132r^5\end{pmatrix}(4R^2+4Rr+3r^2)$$
$$+r\begin{pmatrix}567936R^6+2535072R^5r+2415368R^4r^2+818798R^3r^3+71163R^2r^4\\-13168Rr^5-2044r^6\end{pmatrix}\overset{?}{\geq}0$$
$$\Leftrightarrow 19008t^7+38844t^6+2244t^5-163457t^4-354989t^3+65898t^2+103252t-5720\overset{?}{\geq}0\Leftrightarrow$$
$$(t-2)\left((t-2)\begin{pmatrix}19008t^5+114876t^4+385716t^3+919903t^2\\+1781759t+3513322\end{pmatrix}+7029504\right)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet)$ is true $\therefore$ combining both *cases*, $(\bullet\bullet)\Rightarrow(\bullet)$ is true $\forall\,\Delta\,ABC \therefore \frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c}\geq 3\ \forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equi*lateral* (QED)

1762. In $\Delta\, ABC$ the following relationship holds:

$$a^2\cdot\left(cot\frac{B}{2}+cot\frac{C}{2}\right)+b^2\cdot\left(cot\frac{C}{2}+cot\frac{A}{2}\right)+c^2\cdot\left(cot\frac{A}{2}+cot\frac{B}{2}\right)\geq 72\sqrt{3}r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$a^2\cdot\left(cot\frac{B}{2}+cot\frac{C}{2}\right)+b^2\cdot\left(cot\frac{C}{2}+cot\frac{A}{2}\right)+c^2\cdot\left(cot\frac{A}{2}+cot\frac{B}{2}\right)=$$

$$=a^2\cdot\frac{sin\frac{B+C}{2}}{sin\frac{B}{2}\cdot sin\frac{C}{2}}+b^2\cdot\frac{sin\frac{A+C}{2}}{sin\frac{A}{2}\cdot sin\frac{C}{2}}+c^2\cdot\frac{sin\frac{A+B}{2}}{sin\frac{A}{2}\cdot sin\frac{B}{2}}=$$

$$=a^2\cdot\frac{cos\frac{A}{2}}{sin\frac{B}{2}\cdot sin\frac{C}{2}}+b^2\cdot\frac{cos\frac{B}{2}}{sin\frac{A}{2}\cdot sin\frac{C}{2}}+c^2\cdot\frac{cos\frac{C}{2}}{sin\frac{A}{2}\cdot sin\frac{B}{2}}=$$

$$=\frac{a^2sin\frac{A}{2}\cdot cos\frac{A}{2}+b^2sin\frac{B}{2}+cos\frac{B}{2}+c^2sin\frac{C}{2}\cdot cos\frac{C}{2}}{sin\frac{A}{2}\cdot sin\frac{B}{2}\cdot sin\frac{C}{2}}\geq$$

$$sin\frac{A}{2}\cdot sin\frac{B}{2}\cdot sin\frac{C}{2}\leq\frac{1}{8}\qquad true\ \ (1)$$

$$a=2RsinA\ ;\ b=2RsinB\ ;\ c=2RsinC\ \ (2)$$

$$S=2R^2\cdot sinA\cdot sinB\cdot sinC\qquad(3)$$

$$S\geq 3\sqrt{3}r^2\ \ (4)-Mitrinovic$$

$$\overset{(1)}{\geq}4(a^2sinA+b^2sinB+c^2sinC)\overset{A-G}{\geq}4\cdot 3\cdot\sqrt[3]{(abc)^2sinA\cdot sinB\cdot sinC}\overset{(2)}{=}$$

$$=12\sqrt[3]{64R^6(sinA\cdot sinB\cdot sinC)^3}=48R^2\cdot sinA\cdot sinB\cdot sinC\overset{(3)}{=}48R^2\cdot\frac{S}{2R^2}=$$

$$=24S\overset{(4)}{\geq}72\sqrt{3}r^2$$

Equality holds for a=b=c.

1763.

In any non − right Δ ABC, the following relationship holds :

$$\frac{h_a}{h_b+h_c}.(\sec B+\sec C)+\frac{h_b}{h_c+h_a}.(\sec C+\sec A)+\frac{h_c}{h_a+h_b}.(\sec A+\sec B)\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A,B,C>0,\ (A+B),(B+C),(C+A) \text{ form sides of a triangle}$$
$$(\because (A+B)+(B+C)>(C+A) \text{ and analogs}) \Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A} \text{ form}$$
$$\text{sides of a triangle with area } F \text{ (say) and } 16F^2 =$$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F=2\sqrt{\sum_{cyc}AB} \to (1)$$

$$\text{Now, } \forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have : } \frac{h_a}{h_b+h_c}.(\sec B+\sec C)+\frac{h_b}{h_c+h_a}.(\sec C+\sec A)$$
$$+\frac{h_c}{h_a+h_b}.(\sec A+\sec B) = \frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$(x=h_a, y=h_b, z=h_c, A=\sec A, B=\sec B, C=\sec C)$$
$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$
$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc}\sec A\sec B}$$
$$=\sqrt{3(\sec A\sec B\sec C)\sum_{cyc}\cos A} = \sqrt{3\left(\frac{4R^2}{s^2-4R^2-4Rr-r^2}\right)\left(\frac{R+r}{R}\right)} \overset{\text{Gerretsen}}{\geq}$$
$$\sqrt{3\left(\frac{4R^2}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}\right)\left(\frac{R+r}{R}\right)} = \sqrt{\frac{6R(R+r)}{r^2}} \overset{\text{Euler}}{\geq} \sqrt{\frac{6(2r)(3r)}{r^2}}$$
$$=6 \therefore \frac{h_a}{h_b+h_c}.(\sec B+\sec C)+\frac{h_b}{h_c+h_a}.(\sec C+\sec A)$$
$$+\frac{h_c}{h_a+h_b}.(\sec A+\sec B) \geq 6\ \forall \text{ non}-\text{right } \Delta ABC,$$
$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1764. In ΔABC the following relationship holds:

$$\frac{a}{b\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+c\sin\frac{C}{2}}+\frac{b}{c\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+a\sin\frac{C}{2}}+\frac{c}{a\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+b\sin\frac{C}{2}}\geq 2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$Note:\sum\sin\frac{A}{2}\overset{Jensen}{\leq}3\sin\left(\frac{A+B+C}{6}\right)=3\sin\frac{\pi}{6}=\frac{3}{2}\ (1)$$

$$\frac{a}{b\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+c\sin\frac{C}{2}}+\frac{b}{c\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+a\sin\frac{C}{2}}+\frac{c}{a\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+b\sin\frac{C}{2}}\geq 2$$

$$or\sum\frac{a}{b\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+c\sin\frac{C}{2}}=\sum\frac{a^2}{ab\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+ca\sin\frac{C}{2}}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a+b+c)^2}{(\sum ab)\left(\sum\sin\frac{A}{2}\right)}\overset{3\sum ab\leq(\sum a)^2}{\geq}\frac{(\sum a)^2}{\left(\frac{(\sum a)^2}{3}\right)\left(\frac{3}{2}\right)}\big(using(1)\big)=2$$

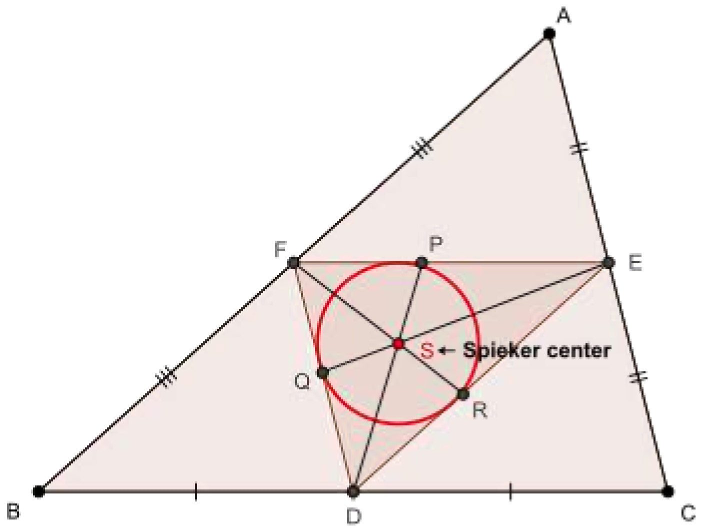
$$Equality\ for\ a=b=c$$

1765.

In any acute Δ ABC with p_a, p_b, p_c → Spieker cevians, the following relationship holds : $r_a(p_a+w_a)+r_b(p_b+w_b)+r_c(p_c+w_c)\leq 2s^2$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and m(∡BAX) = α and m(∡CAX) = β (say)
and inr$a$dius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore\ m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $a$rrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=\mathrm{Rr}\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=\mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$
$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2\mathrm{Rr}\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$
$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$
$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$
$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$
$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$
$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$
$$\text{Also, }p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc) - \left(1 - \frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc) - (2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right) + \left((b+c)^2+2a(b+c)+a^2\right) - a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2 - a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\blacksquare\blacksquare)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \geq w_a \Rightarrow w_a \leq p_a$$

$$\therefore \text{in order to prove}: \frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a}, \text{it suffices to prove}:$$

$$p_a^2 - m_a^2 \leq m_a^2 - w_a^2$$

$$\overset{\text{via }(\blacksquare\blacksquare)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \leq s(s-a) + \frac{(b-c)^2}{4} - \left(s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$= \frac{(b-c)^2}{4}\left(1 + \frac{4s(s-a)}{(2s-a)^2}\right) = \frac{(b-c)^2}{4} \cdot \frac{(2s-a)^2 + 4s(s-a)}{(2s-a)^2}$$

$$\Leftrightarrow \left((2s-a)^2 + 4s(s-a)\right)(2s+a)^2 \geq (8s^2-a^2)(2s-a)^2$$

$$\Leftrightarrow 16s^3 - 12s^2a - 4sa^2 + a^3 \geq 0 \Leftrightarrow (s-a)(16s^2+4sa) + a^3 \geq 0$$

$$\rightarrow \text{true (strict) since } s > a \therefore \frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a} \Rightarrow p_a + w_a \leq 2m_a \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} r_a(p_a+w_a) \leq 2\sum_{cyc} m_a r_a \leq$$

$$2\sum_{cyc} R(1+\cos A)r_a \quad (\because m_a \leq R(1+\cos A) \text{ and analogs } \forall \text{ acute triangles})$$

$$= 2R\sum_{cyc} \frac{2s(s-a)}{bc} \cdot \frac{rs}{s-a} = \frac{4Rrs^2}{4Rrs} \cdot \sum_{cyc} a = 2s^2$$

$$\therefore r_a(p_a+w_a) + r_b(p_b+w_b) + r_c(p_c+w_c) \leq 2s^2$$

$$\forall \text{ acute } \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1766. In ΔABC the following relationship holds:

$$\frac{a}{b(\sin A + \sin B) + c\sin C} + \frac{b}{c(\sin A + \sin B) + a\sin C} + \frac{c}{a(\sin A + \sin B) + a\sin C} \geq \frac{2}{\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum \sin A = \frac{a+b+c}{2R} = \frac{2s}{2R} = \frac{s}{R} \overset{Mitrinovic}{\leq} 3\sqrt{3}\frac{R}{2R} = \frac{3\sqrt{3}}{2} \quad (1)$$

$$\sum \frac{a}{b(\sin A + \sin B) + c\sin C} = \sum \frac{a^2}{ba(\sin A + \sin B) + ca\sin C} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(a+b+c)^2}{(\sum ab)(\sum \sin A)} \overset{3\sum ab \leq (\sum a)^2}{\geq} \frac{(a+b+c)^2}{\frac{(\sum a)^2}{3}(\sum \sin A)} \overset{(1)}{\geq} \frac{3}{\frac{3\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

Equality holds for $a = b = c$

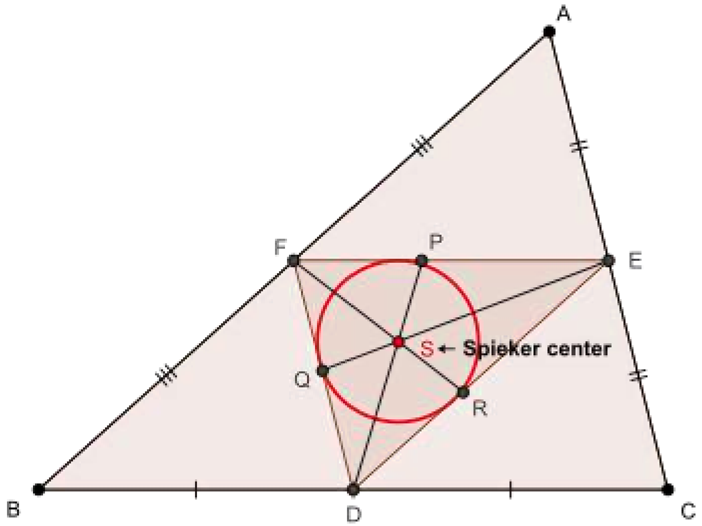
1767.

In any $\Delta\, ABC$ with p_a, p_b, p_c → Spieker cevians, the following relationship holds:

$$(3p_a + w_a).AI + (3p_b + w_b).BI + (3p_c + w_c).CI \geq 2(a^2 + b^2 + c^2)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of $\Delta\, DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\ DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\ \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\ \text{and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Also, }p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)\left(b^3+c^3-abc\right)-\left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right)=\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}$$

$$\therefore\ p_a^2-m_a^2\overset{(\blacksquare\blacksquare)}{=}\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}\overset{s>a}{\geq}0\Rightarrow p_a\geq m_a\Rightarrow p_a w_a\geq m_a w_a\overset{\text{Lascu}+\text{A-G}}{\geq}$$

$$s(s-a)\rightarrow(3)\ \text{Also, }p_a^2-m_a^2\overset{\text{via }(\blacksquare\blacksquare)}{=}\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}\Rightarrow$$

$$p_a^2=s(s-a)+\frac{(b-c)^2}{4}+\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}=s(s-a)+\frac{(b-c)^2\left(12s^2+4sa\right)}{4(2s+a)^2}$$

$$\Rightarrow p_a^2 \overset{(\blacksquare\blacksquare\blacksquare)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } (3p_a+w_a)^2 = 9p_a^2+w_a^2+6p_aw_a \overset{\text{via } (\blacksquare\blacksquare\blacksquare) \text{ and } (3)}{\geq}$$

$$9s(s-a)+\frac{9s(3s+a)(b-c)^2}{(2s+a)^2}+s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}+6s(s-a)$$

$$\overset{?}{\geq} 16m_a^2 = 16s(s-a)+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)(b-c)^2}{(2s-a)^2}+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)+4(2s-a)^2}{(2s-a)^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 20t^4-36t^3+13t^2+5t-2 \overset{?}{\geq} 0 \quad \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\left((t-1)(20t^2+4t+1)+3\right) \overset{?}{\geq} 0 \rightarrow \text{true (strict)} \because t>1$$

$$\therefore 3p_a+w_a \geq 4m_a \text{ and analogs} \Rightarrow \sum_{cyc}\left((3p_a+w_a).AI\right) \geq 4r\sum_{cyc}\left(\frac{m_a}{\sin\frac{A}{2}}\right)$$

$$\overset{Lascu+A-G}{\geq} 4r\sum_{cyc}\left(\frac{\left(\frac{b+c}{2}\cos\frac{A}{2}\right)}{\sin\frac{A}{2}}\right) = 2rs\sum_{cyc}\left(\frac{b+c}{r_a}\right) = \frac{2rs}{rs^2}\sum_{cyc}\left((b+c)s(s-a)\right)$$

$$= 2\sum_{cyc}\left((2s-a)(s-a)\right) = 2\left(2s\sum_{cyc}(s-a)-\left(s(2s)-\sum_{cyc}a^2\right)\right) = 2\sum_{cyc}a^2$$

$$\therefore (3p_a+w_a).AI+(3p_b+w_b).BI+(3p_c+w_c).CI \geq 2(a^2+b^2+c^2)$$

$\forall \Delta ABC, '' ='' $ iff ΔABC is equilateral (QED)

1768. In acute ΔABC holds:

$$h_a(\sec B+\sec C)+h_b(\sec C+\sec A)+h_c(\sec A+\sec A) \geq 36r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$let\ f(x)=\sec x, x\in\left(0,\frac{\pi}{2}\right), f''(x)=\sec x\tan^2 x+\sec^3 x>0,$$

$$so\ f\ is\ convex \in \left(0,\frac{\pi}{2}\right).\ Using\ Jensen\ inequality:$$

$$f(A)+f(B)+f(C) \geq 3f\left(\frac{A+B+C}{3}\right) = 3f\left(\frac{\pi}{3}\right)\ or, \sum \sec A \geq 3\sec\frac{\pi}{3} = 6 \quad (1)$$

$$,\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}, \qquad \sum h_a \overset{AM-GM}{\geq} 3\sqrt[3]{h_ah_bh_c} \overset{Gm-Hm}{\geq} 3.\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}} = 9r \quad (2)$$

$$WLOG\ a\geq b\geq c\ then:$$

$$h_a\leq h_b\leq h_c\ and\ \sec A\geq \sec B\geq \sec C\ and$$

$$\sec A+\sec B\geq \sec A+\sec C\geq \sec C+\sec B$$

$$h_a(\sec B+\sec C)+h_b(\sec C+\sec A)+h_c(\sec A+\sec A)\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\left(\sum h_a\right)\left(\sum \sec A+\sec B\right)=\frac{1}{3}\left(\sum h_a\right)\left(2\sum \sec A\right)\overset{(1)\&(2)}{\geq}$$

$$\geq\frac{1}{3}.9r.2.6=36r$$

$$Equality\ for\ a=b=c$$

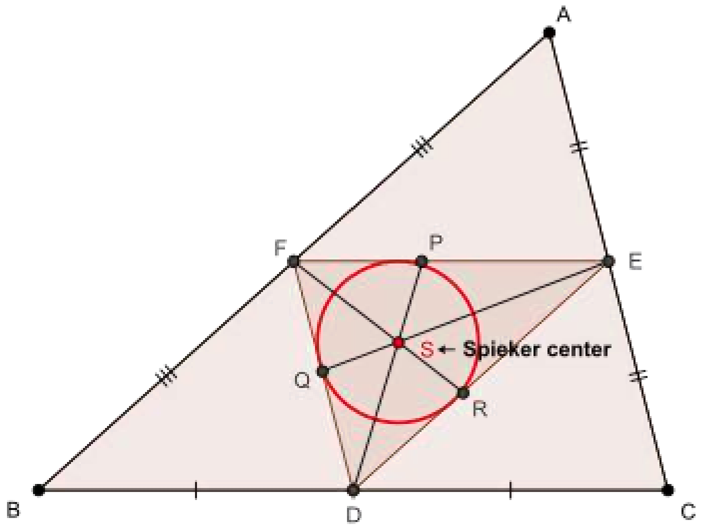
1769.

In any Δ ABC with p_a, p_b, p_c
$\rightarrow$ Spieker cevians, the following relationship holds :

$$(3p_a+w_a)\sin\frac{A}{2}+(3p_b+w_b)\sin\frac{B}{2}+(3p_c+w_c)\sin\frac{C}{2}\geq 2(h_a+h_b+h_c)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say)
and inradius of Δ DEF $=r'$(say)

$$\text{Now, }16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [\mathrm{DEF}]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$\because$ **Spieker center is incenter of Δ DEF,** $\therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2}\ \text{and}\ m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\text{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\text{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\big(b^3+c^3-abc+a(4m_a^2)\big)}$$

$$\text{Also, }p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\big(b^3+c^3-abc+a(4m_a^2)\big)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\Big(\big(a^2+2a(b+c)+(b+c)^2\big)+\big((b+c)^2+2a(b+c)+a^2\big)-a^2\Big)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\big(2(a+b+c)^2-a^2\big)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\blacksquare\blacksquare)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\overset{s>a}{\geq}0\Rightarrow p_a\geq m_a\Rightarrow p_a w_a\geq m_a w_a\overset{\text{Lascu + A-G}}{\geq}$$

$$s(s-a)\to(3)\text{ Also, }p_a^2-m_a^2\overset{\text{via }(\blacksquare\blacksquare)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\Rightarrow$$

$$p_a^2=s(s-a)+\frac{(b-c)^2}{4}+\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}=s(s-a)+\frac{(b-c)^2(12s^2+4sa)}{4(2s+a)^2}$$

$$\Rightarrow p_a^2\overset{(\blacksquare\blacksquare\blacksquare)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } (3p_a+w_a)^2 = 9p_a^2+w_a^2+6p_aw_a \overset{\text{via } (\blacksquare\blacksquare\blacksquare) \text{ and } (3)}{\geq}$$

$$9s(s-a)+\frac{9s(3s+a)(b-c)^2}{(2s+a)^2}+s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}+6s(s-a)$$

$$\overset{?}{\geq} 16m_a^2 = 16s(s-a)+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)(b-c)^2}{(2s-a)^2}+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)+4(2s-a)^2}{(2s-a)^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 20t^4-36t^3+13t^2+5t-2 \overset{?}{\geq} 0 \quad \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\left((t-1)(20t^2+4t+1)+3\right) \overset{?}{\geq} 0 \to \text{true (strict)} \because t>1$$

$$\therefore 3p_a+w_a \geq 4m_a \text{ and analogs} \Rightarrow \sum_{cyc}\left((3p_a+w_a)\sin\frac{A}{2}\right) \geq 4\sum_{cyc}\left(m_a\sin\frac{A}{2}\right)$$

$$\overset{\text{Lascu}}{\geq} 4\sum_{cyc}\left(\left(\frac{b+c}{2}\cos\frac{A}{2}\right)\sin\frac{A}{2}\right) = \sum_{cyc}\left((b+c).\frac{a}{2R}\right) = 2.\sum_{cyc}\frac{bc}{2R} = 2\sum_{cyc}h_a$$

$$\therefore (3p_a+w_a)\sin\frac{A}{2}+(3p_b+w_b)\sin\frac{B}{2}+(3p_c+w_c)\sin\frac{C}{2} \geq 2(h_a+h_b+h_c)$$

$$\forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1770. In any acute ΔABC the following relationship holds:

$$a(\sec B+\sec C)+b(\sec C+\sec A)+c(\sec B+\sec C) \geq 24\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum\cos A = 1+\frac{r}{R} \overset{\text{Euler}}{\leq} \frac{3}{2}, \prod\cos A \overset{\text{AM-GM}}{\leq} \left(\frac{\sum\cos A}{3}\right)^3 \leq \left(\frac{3}{2}.\frac{1}{3}\right)^3 = \frac{1}{8}$$

$$\text{so} \prod\sec A \geq 8 \quad (1)$$

$$a(\sec B+\sec C)+b(\sec C+\sec A)+c(\sec B+\sec C) =$$

$$= \sum a(\sec B+\sec C) \overset{\text{AM-GM}}{\geq}$$

$$2\sum a\sqrt{\sec B\sec C} \overset{AM-GM}{\geq} 6\sqrt[3]{abc\sec B\sec C\sec A} \overset{(1)}{\geq}$$

$$\geq 6\sqrt[3]{4Rrs.8}\overset{Euler\ \&\ Mritrinovic}{\geq}6\left(4.2r.r.3\sqrt{3}r.8\right)^{\frac{1}{3}}=24\sqrt{3}\,r$$

$$Equality\ holds\ \ for\ a=b=c$$

1771. In ΔABC the following relationship holds:

$$\frac{h_a^n}{h_b^n+h_c^n}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq\sqrt{3},\qquad n\in\mathbb{N}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ ,then$$

$$h_a\leq h_b\leq h_c\ and\ \ h_a+h_b\leq h_a+h_c\leq h_b+h_c\ ,$$

$$\frac{h_a^n}{h_b^n+h_c^n}\leq\frac{h_b^n}{h_c^n+h_a^n}\leq\frac{h_c^n}{h_a^n+h_b^n}$$

$$and\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\leq\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\leq\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$$

$$\frac{h_a^n}{h_b^n+h_c^n}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq$$

$$\overset{Chebyshev}{\geq}\frac{1}{3}\cdot\sum\frac{h_a^n}{h_b^n+h_c^n}\sum\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\overset{Nesbitt}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{3}{2}\cdot\frac{2(4R+r)}{s}\geq\sqrt{3}\ \left(since\,\frac{4R+r}{s}\geq\sqrt{3}\right)$$

Equality for a=b=c

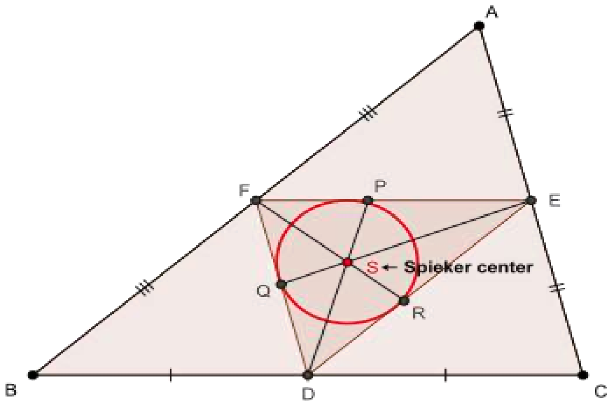
1772.

In any Δ ABC with p_a, p_b, p_c $\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{p_a}{b+c}+\frac{p_b}{c+a}+\frac{p_c}{a+b}\leq\frac{s}{4r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC at X and } m(\measuredangle BAX) = \alpha \text{ and } m(\measuredangle CAX) = \beta \text{ (say)}$$
$$\text{and inradius of } \Delta\, DEF = r' \text{ (say)}$$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$\text{Via (1), (2) and using cosine law on } \Delta\, AFS \text{ and } \Delta\, AES, \text{ we arrive at :}$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3+a^3-abc+a\left(2b^2+2c^2-a^2\right)-a^3$$

$$= \sum_{cyc} a^3-abc+2a.2bc\cos A = 2s\left(s^2-6Rr-3r^2\right)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2)\overset{(\blacksquare\blacksquare)}{=}2s(s^2-8Rr-3r^2+8Rr\cos A)$$

$$\text{We have}: \prod_{cyc}(2s+a)=8s^3+4s^2\sum_{cyc}a+2s\sum_{cyc}ab+4Rrs$$

$$=8s^3+4s^2.2s+2s(s^2+4Rr+r^2)+4Rrs$$

$$\Rightarrow \prod_{cyc}(2s+a)\overset{(\blacksquare\blacksquare\blacksquare)}{=}2s(9s^2+6Rr+r^2)\text{ and}$$

$$\sum_{cyc}(2s+b)(2s+c)=\sum_{cyc}(4s^2+2s(2s-a)+bc)$$

$$=24s^2-2s(2s)+s^2+4Rr+r^2\Rightarrow\sum_{cyc}(2s+b)(2s+c)\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}21s^2+4Rr+r^2$$

$$(\blacksquare),(\blacksquare\blacksquare)\Rightarrow p_a=\frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}\overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=}$$

$$\frac{2s}{2s(9s^2+6Rr+r^2)}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}.(2s+b)(2s+c)\text{ and analogs}$$

$$\Rightarrow p_a+p_b+p_c$$

$$=\frac{1}{9s^2+6Rr+r^2}.\sum_{cyc}\left(\sqrt{(s^2-8Rr-3r^2+8Rr\cos A)(2s+b)(2s+c)}.\sqrt{(2s+b)(2s+c)}\right)$$

$$\overset{CBS}{\leq}\frac{1}{9s^2+6Rr+r^2}.\sqrt{\sum_{cyc}(s^2-8Rr-3r^2+8Rr\cos A)(2s+b)(2s+c)}.\sqrt{\sum_{cyc}(2s+b)(2s+c)}$$

$$\overset{\text{via }(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}\frac{1}{9s^2+6Rr+r^2}.\sqrt{\begin{matrix}(s^2-8Rr-3r^2)(21s^2+4Rr+r^2)+\\ 8Rr\sum_{cyc}((8s^2-2sa+bc)\cos A)\end{matrix}}.\sqrt{21s^2+4Rr+r^2}$$

$$=\frac{1}{9s^2+6Rr+r^2}.\sqrt{\begin{matrix}(s^2-8Rr-3r^2)(21s^2+4Rr+r^2)\\ +8Rr\left(8s^2.\frac{R+r}{R}-2s.\frac{2rs}{R}+\sum_{cyc}\left(bc.\frac{b^2+c^2-a^2}{2bc}\right)\right)\end{matrix}}.\sqrt{21s^2+4Rr+r^2}$$

$$\left(\because\sum_{cyc}a\cos A=\frac{2rs}{R}\right)$$

$$=\frac{1}{9s^2+6Rr+r^2}.\sqrt{\begin{matrix}(s^2-8Rr-3r^2)(21s^2+4Rr+r^2)\\ +8r(8(R+r)s^2-4rs^2+R(s^2-4Rr-r^2))\end{matrix}}.\sqrt{21s^2+4Rr+r^2}$$

$$\Rightarrow(p_a+p_b+p_c)^2\leq$$

$$\frac{(21s^2+4Rr+r^2)\left(21s^4-(92Rr+30r^2)s^2-r^2(64R^2+28Rr+3r^2)\right)}{(9s^2+6Rr+r^2)^2}\overset{?}{\leq}\frac{(14R-r)^2}{9}$$

$$\Leftrightarrow\boxed{\begin{matrix}3969s^6-(15876R^2+14364Rr+5562)s^4\\ -rs^2(21168R^3+15912R^2r+6804Rr^2+855r^3)\\ -r^2(7056R^4+3648R^3r+1480R^2r^2+344Rr^3+28r^4)\overset{?}{\underset{(\bullet)}{\leq}}0\end{matrix}}$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\ge 0 \text{ and } s^2-(m+n)\le 0, \text{ where}$$
$$m=2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$
$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0 \Rightarrow s^4-s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3 \overset{(\blacklozenge)}{\le} 0$$
$$\therefore 3969s^2\left(s^4-\left(4R^2+20Rr-2r^2\right)s^2+r(4R+r)^3\right)\le 0 \Rightarrow \text{in order}$$
$$\text{to prove } (\bullet), \text{it suffices to prove : LHS of } (\bullet)\le$$
$$3969s^2\left(s^4-\left(4R^2+20Rr-2r^2\right)s^2+r(4R+r)^3\right)$$

$$\Leftrightarrow \boxed{\begin{gathered}(16254R-3375r)s^4-s^2\left(68796R^3+51606R^2r+13608Rr^2+1206r^2\right)\\ -r\left(1764R^4+912R^3r+370R^2r^2+86Rr^3+7r^4\right)\underset{(\bullet\bullet)}{\overset{?}{\le}} 0\end{gathered}} \text{ and}$$

$$\because (16254R-3375r)\left(s^4-\left(4R^2+20Rr-2r^2\right)s^2+r(4R+r)^3\right) \overset{\text{via }(\blacklozenge)}{\le} 0$$
$$\therefore \text{in order to prove } (\bullet\bullet), \text{it suffices to prove : LHS of } (\bullet\bullet)\le$$
$$(16254R-3375r)\left(s^4-\left(4R^2+20Rr-2r^2\right)s^2+r(4R+r)^3\right)$$

$$\Leftrightarrow \boxed{\begin{gathered}\left(1890R^3-129987R^2r+56808Rr^2-2772r^3\right)s^2\\ +r\left(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4\right)\overset{(\bullet\bullet\bullet)}{\ge} 0\end{gathered}}$$

$\boxed{Case\ 1}$ $1890R^3-129987R^2r+56808Rr^2-2772r^3\ge 0$ *and* then : LHS of $(\bullet\bullet\bullet)$
$$\ge r\left(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4\right)>0$$
$$\Rightarrow (\bullet\bullet\bullet) \text{ is true}$$

$\boxed{Case\ 2}$ $1890R^3-129987R^2r+56808Rr^2-2772r^3<0$ *and* then : LHS of $(\bullet\bullet\bullet)$
$$=-\left(-\left(1890R^3-129987R^2r+56808Rr^2-2772r^3\right)\right)s^2$$
$$+r\left(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4\right)$$
$$\overset{\text{Gerretsen}}{\ge} -\left(-\left(1890R^3-129987R^2r+56808Rr^2-2772r^3\right)\right)\left(4R^2+4Rr+3r^2\right)$$
$$+r\left(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4\right)\overset{?}{\ge} 0$$
$$\Leftrightarrow 3780t^5+1287t^4-8439t^3-85538t^2+65924t-3704\overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left(3780t^4+11871t^3+3713t^2+17782t(t-2)+2500\right)\overset{?}{\ge} 0 \to \text{true}$$
$$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\bullet\bullet\bullet) \text{ is true} \therefore \text{combining both cases}, (\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)$$
$$\text{is true } \forall\, \Delta\, ABC \Rightarrow (p_a+p_b+p_c)^2\le \frac{(14R-r)^2}{9}$$
$$\therefore \boxed{p_a+p_b+p_c\le \frac{14R-r}{3}} \therefore \sum_{cyc}\frac{p_a}{b+c}$$
$$=\frac{1}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left(p_a(c+a)(a+b)\right)$$
$$=\frac{1}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left(p_a\left(\sum_{cyc}ab\right)+a^2p_a\right)$$

$$\le \frac{1}{2s(s^2+2Rr+r^2)}\cdot\left(\frac{14R-r}{3}\cdot\left(\sum_{cyc} ab\right)+\sum_{cyc} a^2 p_a\right)$$

$$\overset{\text{Chebyshev}}{\le} \frac{1}{2s(s^2+2Rr+r^2)}\cdot\left(\frac{14R-r}{3}\cdot\left(\sum_{cyc} ab\right)+\frac{1}{3}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} p_a\right)\right)$$

$$\left(\begin{array}{c}\because p_a=\frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}} \Rightarrow \text{in order to prove : } p_a \le p_b \text{ for } a \ge b,\\ \text{it suffices to prove : } \frac{1}{2s+a}\le\frac{1}{2s+b} \text{ and}\\ s^2-3r^2-16Rr\sin^2\frac{A}{2}\le s^2-3r^2-16Rr\sin^2\frac{B}{2}, \text{both of which are true}\\ \therefore \text{WLOG assuming } a\ge b\ge c \Rightarrow p_a\le p_b\le p_c \text{ and } a^2\ge b^2\ge c^2\end{array}\right)$$

$$\le \frac{1}{2s(s^2+2Rr+r^2)}\cdot\left(\frac{14R-r}{3}\cdot\left(\sum_{cyc} ab\right)+\frac{1}{3}\left(\sum_{cyc} a^2\right)\left(\frac{14R-r}{3}\right)\right)$$

$$=\frac{\frac{14R-r}{9}}{2s(s^2+2Rr+r^2)}\left(\left(2\sum_{cyc} ab+\sum_{cyc} a^2\right)+\sum_{cyc} ab\right)$$

$$=\frac{(14R-r)(4s^2+s^2+4Rr+r^2)}{18s(s^2+2Rr+r^2)}\overset{?}{\le}\frac{s}{4r}$$

$$\Leftrightarrow 9s^4-(122Rr-19r^2)s^2-r^2(112R^2+20Rr-2r^2)\underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\ge}} 0$$

$$\text{Now, LHS of } (\bullet\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\ge} (144Rr-45r^2)s^2-(122Rr-19r^2)s^2$$
$$-r^2(112R^2+20Rr-2r^2)=(22Rr-26r^2)s^2-r^2(112R^2+20Rr-2r^2)$$
$$\overset{\text{Gerretsen}}{\ge} (22Rr-26r^2)(16Rr-5r^2)-r^2(112R^2+20Rr-2r^2)\overset{?}{\ge}0$$
$$\Leftrightarrow 40R^2-91Rr+22r^2\overset{?}{\ge}0 \Leftrightarrow (40R-11r)(R-2r)\overset{?}{\ge}0 \to \text{true} \because R\overset{\text{Euler}}{\ge}2r$$
$$\Rightarrow (\bullet\bullet\bullet\bullet) \text{ is true} \therefore \frac{p_a}{b+c}+\frac{p_b}{c+a}+\frac{p_c}{a+b}\le\frac{s}{4r}$$
$$\forall\,\Delta\,ABC, \text{''}=\text{'' iff } \Delta\,ABC \text{ is equilateral (QED)}$$

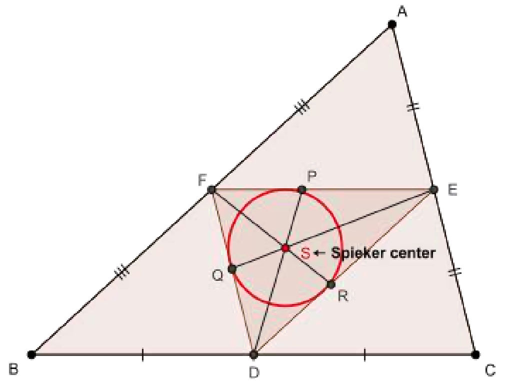
1773.

In any acute Δ ABC with p_a → Spieker cevian, the following relationship holds :

$$p_a \ge \sqrt{\frac{b^2+c^2}{2}}\cdot\cos\frac{A}{2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say)
and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore\ m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2\mathrm{Rr}$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4} - 2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2=s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\ \therefore p_a\ge\sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2}$$

$$\Leftrightarrow s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\ge\frac{b^2+c^2}{2}.\frac{s(s-a)}{bc}$$

$$\Leftrightarrow\frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}}{s(s-a)}-1\ge\frac{b^2+c^2}{2bc}-1\Leftrightarrow\frac{s(3s+a)(b-c)^2}{s(s-a)(2s+a)^2}\ge\frac{(b-c)^2}{2bc}$$

$$\Leftrightarrow\frac{s(3s+a)}{s(s-a)(2s+a)^2}\ge\frac{1}{2bc}\ (\because (b-c)^2\ge 0)\Leftrightarrow\frac{2s(3s+a)}{(2s+a)^2}\ge\frac{s(s-a)}{bc}=\cos^2\frac{A}{2}$$

$$\Leftrightarrow\frac{2s(3s+a)}{(2s+a)^2}-1\ge\cos^2\frac{A}{2}-1\Leftrightarrow\boxed{\frac{2s^2-2sa-a^2}{(2s+a)^2}+\sin^2\frac{A}{2}\overset{(\blacklozenge)}{\ge}0}$$

$$\text{Now, } \sin\frac{A}{2} = \frac{\sin A}{2\cos\frac{A}{2}} \overset{0<\cos\frac{A}{2}<1}{>} \frac{\sin A}{2} = \frac{a}{4R} > \frac{a}{2s}$$

$$\left(\because \Delta\, ABC \text{ being acute} \Rightarrow \prod_{cyc} \cos A = \frac{s^2 - (2R+r)^2}{4R^2} > 0 \Rightarrow s > 2R + r > 2R\right)$$

$$\Rightarrow \sin^2\frac{A}{2} > \frac{a^2}{4s^2} \Rightarrow \text{LHS of } (\blacklozenge) > \frac{2s^2 - 2sa - a^2}{(2s+a)^2} + \frac{a^2}{4s^2} = \frac{8s^4 - 8s^3a + 4sa^3 + a^4}{4s^2(2s+a)^2}$$

$$= \frac{8s^3(s-a) + 4sa^3 + a^4}{4s^2(2s+a)^2} \overset{s>a}{>} 0 \Rightarrow (\blacklozenge) \text{ is true (strict inequality)}$$

$$\therefore p_a \geq \sqrt{\frac{b^2+c^2}{2}}\,.\cos\frac{A}{2}\ \forall \text{ acute } \Delta\, ABC, "=" \text{ iff } b = c \text{ (QED)}$$

1774.

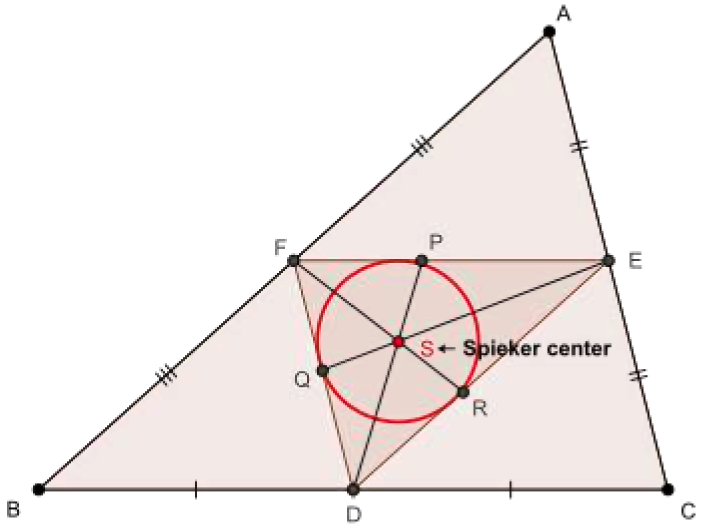
In any Δ ABC with

$p_a \to$ Spieker cevian, the following relationship holds :

$$p_a \leq \frac{(b+c)^2}{16r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF = r′ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1-2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathbf{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{Now, }b^3+c^3-abc+a\big(4m_a^2\big)=b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$=(b+c)\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2\big)+a\big(b^2+c^2-a^2\big)$$

$$=2s\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2+bc-a^2\big)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Now, $p_a \le \frac{(b+c)^2}{16r} \Leftrightarrow p_a.\frac{2rs}{a} \le \frac{(b+c)^2}{8}.\frac{s}{a} \Leftrightarrow p_a h_a \le \frac{s(b+c)^2}{8a} \Leftrightarrow$

$$p_a^2 h_a^2 \le \frac{s^2(b+c)^4}{64a^2} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)\le\frac{s^2(b+c)^4}{64a^2}$$

$$\Leftrightarrow s^2(s-a)^2-\frac{s^2(s-a)^2(b-c)^2}{a^2}+\frac{s(3s+a).s(s-a)(b-c)^2}{(2s+a)^2}$$
$$-\frac{s(3s+a).s(s-a)(b-c)^4}{a^2(2s+a)^2}\le\frac{s^2(b+c)^4}{64a^2}$$

$$\Leftrightarrow \frac{(2s-a)^4}{64a^2}-(s-a)^2+(s-a)(b-c)^2\left(\frac{s-a}{a^2}-\frac{3s+a}{(2s+a)^2}\right)$$
$$+\frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2}\ge 0$$

$$\Leftrightarrow \frac{\left((2s-a)^2-8a(s-a)\right)\left((2s-a)^2+8a(s-a)\right)}{64a^2}$$
$$+\frac{(s-a)(b-c)^2\left((s-a)(2s+a)^2-a^2(3s+a)\right)}{a^2(2s+a)^2}+\frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2}\ge 0$$

$$\Leftrightarrow \boxed{\frac{(2s-3a)^2(4s^2+4sa-7a^2)}{64a^2}+\frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2}+\frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2}\overset{(\blacklozenge)}{\ge} 0}$$

Case 1 $(2s-3a)^2 \ge (b-c)^2$ and then : LHS of $(\blacklozenge) \ge$

$$\frac{(4s^2+4sa-7a^2)(b-c)^2}{64a^2}+\frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2}\overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{4s^2+4sa-7a^2}{64}+\frac{2(s^2-a^2)(2s^2-2sa-a^2)}{(2s+a)^2} \overset{?}{\geq} 0\ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 272s^4-224s^3a-392s^2a^2+232sa^3+121a^4 \boxed{\underset{(\blacksquare)}{\overset{?}{\geq}}} 0 \text{ and}$$

$$\because 272(s-a)^4+864a(s-a)^3>0 \therefore \text{in order to prove } (\blacksquare),$$

$$\text{it suffices to prove : LHS of } (\blacksquare) > 272(s-a)^4+864a(s-a)^3$$

$$\Leftrightarrow 568t^2-1272t+713>0\ \left(t=\frac{s}{a}\right), \text{which is true} \because \text{discriminant}$$

$$=1272^2-4(568)(713)=-1952<0 \Rightarrow (\blacksquare) \Rightarrow (\blacklozenge) \text{ is true}$$

$$\boxed{\text{Case 2}}\ (b-c)^2 \geq (2s-3a)^2 \text{ and then : LHS of } (\blacklozenge) \geq$$

$$\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2}+\frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2}$$

$$+\frac{(3s+a)(s-a)(b-c)^2(2s-3a)^2}{a^2(2s+a)^2}=\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2}$$

$$+\frac{(s-a)(b-c)^2}{a^2(2s+a)^2}.\left(2(s+a)(2s^2-2sa-a^2)+(3s+a)(2s-3a)^2\right)$$

$$=\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2}+\frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2}.(16s^2-16sa-7a^2)$$

$$\text{which is definitely} \geq 0 \text{ if : } 16s^2-16sa-7a^2 \geq 0 \text{ and so, we now consider :}$$

$$16s^2-16sa-7a^2<0 \text{ and} \because (b-c)^2<a^2 \therefore \text{LHS of } (\blacklozenge) \geq$$

$$\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2}+\frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2}.(16s^2-16sa-7a^2)$$

$$>\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2}+\frac{(s-a)^2(16s^2-16sa-7a^2)}{(2s+a)^2} \overset{?}{>} 0 \Leftrightarrow$$

$$64s^6-64s^5a+752s^4a^2-2784s^3a^3+2812s^2a^4-260sa^5-511a^6 \boxed{\underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}}} 0 \text{ and}$$

$$\because (4s-5a)^6+104a(s-a)(4s-5a)^4+437a^3(4s-5a)^4>0$$

$$\therefore \text{in order to prove } (\blacksquare\blacksquare), \text{it suffices to prove :}$$

$$64\bullet\text{LHS of } (\blacksquare\blacksquare) > (4s-5a)^6+104a(s-a)(4s-5a)^4+437a^3(4s-5a)^4$$

$$\Leftrightarrow 79232t^3-280616t^2+329680t-128227>0$$

$$\Leftrightarrow (t-1)(79232t^2-201384t+128296)+69>0 \rightarrow \text{true} \because t=\frac{s}{a}>1 \text{ and}$$

$$\because \text{discriminant of } (79232t^2-201384t+128296)$$

$$=201384^2-4(79232)(128296)=-105079232$$

$$\Rightarrow 79232t^2-201384t+128296>0 \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacklozenge) \text{ is true}$$

$$\therefore \text{combining both cases}, (\blacklozenge) \text{ is true } \forall\, \Delta\, ABC \therefore p_a \leq \frac{(b+c)^2}{16r}$$

$$\forall\, \Delta\, ABC, "=" \text{ iff } 2s-3a=0 \text{ and } b=c \Rightarrow "=" \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1775. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{b(\sin^2 A+\sin^2 B)+c\sin^2 C}\ge\frac{4}{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum\sin^2 A=\left(\sum\sin A\right)^2-2\sum\sin A\sin B=\left(\frac{s}{R}\right)^2-\frac{2(s^2+r^2+4Rr)}{4R^2}=$$

$$=\frac{2s^2-2r^2-8Rr}{4R^2}\overset{Gerretsen}{\le}\frac{8R^2+4r^2}{4R^2}=2+\left(\frac{r}{R}\right)^2\overset{Euler}{\le}\frac{9}{4}\ (1)$$

$$\sum\frac{a}{b(\sin^2 A+\sin^2 B)+c\sin^2 C}=\sum\frac{a^2.}{ba(\sin^2 A+\sin^2 B)+ca\sin^2 C}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(\sum a)^2}{(\sum ab)(\sum\sin^2 A)}\overset{3\sum ab\le(\sum a)^2\&(1)}{\ge}\frac{(\sum a)^2}{\frac{(\sum a)^2}{3}\frac{9}{4}}=\frac{4}{3}$$

$$Equality\ for\ a=b=c.$$

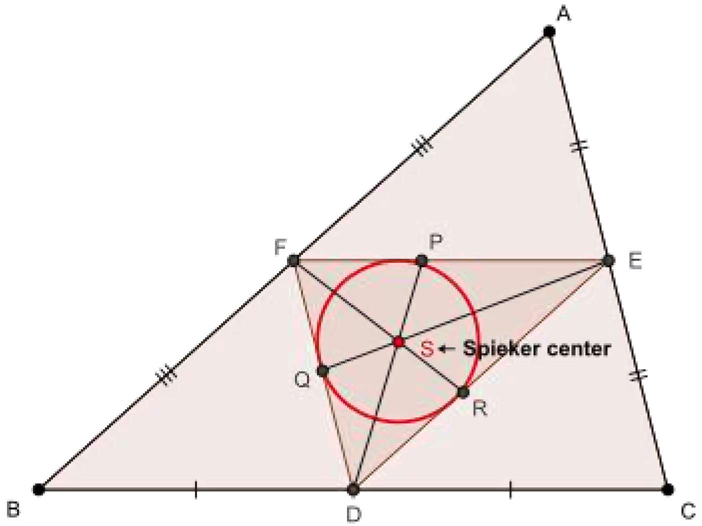
1776.

In any Δ ABC with p_a, p_b, p_c $\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{3p_a-2m_a}{h_a}+\frac{3p_b-2m_b}{h_b}+\frac{3p_c-2m_c}{h_c}\le\frac{2R}{r}-1$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC } at \text{ X } and \text{ m}(\measuredangle BAX) = \alpha \ and \ \text{m}(\measuredangle CAX) = \beta \ (say)$$
$$and \text{ inradius of } \Delta \text{ DEF} = r' (say)$$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore \text{ m}(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$
$$= \frac{\pi}{2} - \frac{A-B}{2} \ and \ \text{m}(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$\text{Via } (1),(2) \ and \text{ using cosine law on } \Delta \text{ AFS } and \ \Delta \text{ AES, we arrive at :}$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$
$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$

$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$= 2s\left(b^2 - bc + c^2\right) + a\left(b^2 - bc + c^2 + bc - a^2\right)$$

$$= (2s + a)\left(b^2 - bc + c^2\right) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right)$$

$$= (2s + a)\left(b^2 - bc + c^2\right) + \frac{a(b + c + 2a)(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(b^2 - bc + c^2\right) + \frac{a(2s - a + 2a)(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a).\frac{4b^2 + 4c^2 - 4bc + a(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a).$$

$$\frac{4(z + x)^2 + 4(x + y)^2 - 4(z + x)(x + y) + (y + z)\left((z + x) + (x + y) - 2(y + z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a = y + z, b = z + x, c = x + y)$$

$$= (2s + a).\frac{4x(x + y + z) + 2x(y + z) + 3(y - z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{3}{4}(b - c)^2 + \frac{a(s - a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{3}{4}(b - c)^2 + \frac{a(s - a)}{2}\right) - \frac{(a + 2s - 2s)(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{(b-c)^2}{2} + \frac{a(s - a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3 + c^3 - abc + a\left(4m_a^2\right) \overset{(\bullet\bullet)}{=} (2s + a)\left(\frac{(s - a)(2s + a)}{2} + \frac{(b-c)^2}{2}\right) + \frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s + a)^2}\left(\frac{(s - a)(2s + a)^2}{2} + \frac{(2s + a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s - a) + (b - c)^2\left(\left(\frac{s}{2s + a}\right)^2 + \frac{s}{2s + a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s - a) - \frac{(b-c)^2}{4} + (b - c)^2.\left(\frac{s}{2s + a} + \frac{1}{2}\right)^2$$

$$= s(s - a) + \frac{(b-c)^2}{4}\left(\frac{(4s + a)^2}{(2s + a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s - a) + \frac{s(3s + a)(b-c)^2}{(2s + a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s - a) + \frac{(b-c)^2}{4}\right)\left(s(s - a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s - a) + \frac{s(3s + a)(b-c)^2}{(2s + a)^2}\right)^2$$

$$+ \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}.\left(s(s - a) + \frac{s(3s + a)(b-c)^2}{(2s + a)^2}\right)$$

$$\Leftrightarrow s(s - a)(b - c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s + a)^2(b-c)^4}{(2s + a)^4} +$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow \frac{s(s-a)\left(144s^3-52s^2a-16sa^2+5a^3\right)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)\left(144s^2+92sa+76a^2\right)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)\left(1296s^3+1820s^2a+1736sa^2+1700a^3\right)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq}0 \to \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2+\frac{(b-c)^2}{18}\to (3)$$

$$\text{Now, } (2m_a+n_a)^2-9p_a^2=4m_a^2+n_a^2+4m_a n_a-9p_a^2\overset{\text{via }(3)}{\geq}$$

$$4s(s-a)+(b-c)^2+s(s-a)+\frac{s(b-c)^2}{a}+4p_a^2+\frac{2(b-c)^2}{9}-9p_a^2$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{=}5s(s-a)+\frac{11(b-c)^2}{9}+\frac{s(b-c)^2}{a}-5s(s-a)-\frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$=\left(\frac{11}{9}+\frac{s}{a}-\frac{5s(3s+a)}{(2s+a)^2}\right).(b-c)^2=\frac{36s^3-55s^2a+8sa^2+11a^3}{9a(2s+a)^2}.(b-c)^2$$

$$=\frac{(s-a)\left((s-a)(36s+17a)+6a^2\right)}{9a(2s+a)^2}.(b-c)^2\geq 0 \therefore (2m_a+n_a)^2\geq 9p_a^2$$

$$\Rightarrow 2m_a+n_a\geq 3p_a \Rightarrow \frac{3p_a-2m_a}{h_a}\leq\frac{n_a}{h_a}\ \textit{and analogs}\to (4)$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s\left(b^2+c^2\right)-2sbc$$

$$=an_a^2+a\left(as-s^2\right)\Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2\Rightarrow an_a^2=$$

$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_a r_a \therefore n_a^2=s^2-2r_a h_a$$

$$\therefore \sum_{cyc}\frac{3p_a-2m_a}{h_a}\overset{\text{via }(4)}{\leq}\sum_{cyc}\frac{n_a}{h_a}=\frac{1}{2rs}\sum_{cyc}an_a\overset{\text{CBS}}{\leq}\frac{1}{2rs}.\sqrt{\sum_{cyc}a}.\sqrt{\sum_{cyc}an_a^2}$$

$$=\frac{1}{2rs}.\sqrt{2s}.\sqrt{\sum_{cyc}a\left(s^2-2r_a h_a\right)}=\frac{1}{2rs}.\sqrt{2s}.\sqrt{2s^3-4rs\sum_{cyc}r_a}=\frac{\sqrt{s^2-8Rr-2r^2}}{r}$$

$$\overset{\text{Gerretsen}}{\leq}\frac{\sqrt{4R^2+4Rr+3r^2-8Rr-2r^2}}{r}=\frac{\sqrt{(2R-r)^2}}{r}$$

$$\therefore \frac{3p_a - 2m_a}{h_a} + \frac{3p_b - 2m_b}{h_b} + \frac{3p_c - 2m_c}{h_c} \le \frac{2R}{r} - 1$$

$\forall \Delta ABC, " = "$ iff ΔABC is equilateral (QED)

1777. In ΔABC the following relationship holds:

$$\frac{a}{b\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+c\cos\frac{C}{2}} + \frac{b}{c\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+a\cos\frac{C}{2}} + \frac{c}{a\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+a\cos\frac{C}{2}} \ge \frac{2}{\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum \cos\frac{A}{2} = \sqrt{\left(\sum \cos\frac{A}{2}\right)^2} \le \sqrt{3\sum\left(\cos^2\frac{A}{2}\right)} = \sqrt{3\left(2+\frac{r}{2R}\right)} \overset{Euler}{\le} \sqrt{3.\left(2+\frac{1}{4}\right)} = \frac{3\sqrt{3}}{2} \quad (1)$$

$$\frac{a}{b\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+c\cos\frac{C}{2}} + \frac{b}{c\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+a\cos\frac{C}{2}} + \frac{c}{a\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+a\cos\frac{C}{2}} =$$

$$= \sum \frac{a}{b\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+c\cos\frac{C}{2}} = \sum \frac{a^2}{ba\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)+ca\cos\frac{C}{2}} \overset{Bergstrom}{\ge}$$

$$\ge \frac{(\sum a)^2}{(\sum ab)\left(\sum\cos\frac{A}{2}\right)} \overset{(1)}{\ge} \frac{(\sum a)^2}{\frac{(\sum a)^2}{3}\left(\frac{3\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}}$$

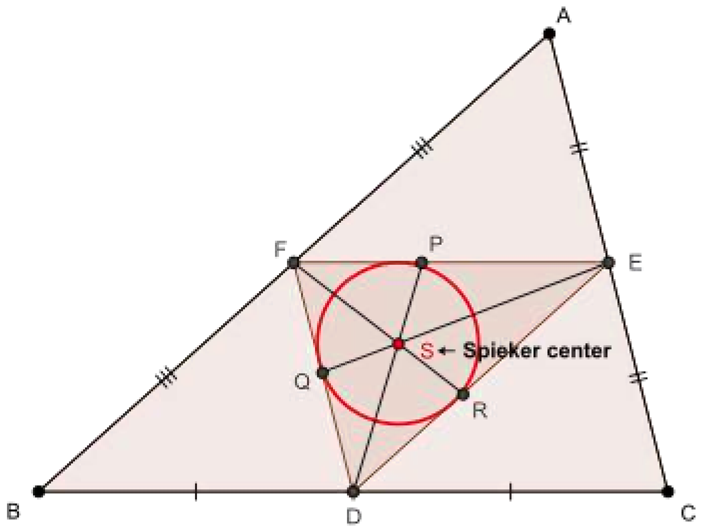
Equality holds for $a = b = c$

1778.

In any Δ ABC with p_a, n_a, g_a
$\rightarrow$ Spieker cevian, Nagel cevian, Gergonne cevian,
the following relationship holds :
$$w_a + 2p_a \le 4m_a - w_a \le 3p_a \le 2m_a + n_a \le g_a + 2n_a$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES,\ b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$

$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$= 2s\left(b^2 - bc + c^2\right) + a\left(b^2 - bc + c^2 + bc - a^2\right)$$

$$= (2s + a)\left(b^2 - bc + c^2\right) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right)$$

$$= (2s + a)\left(b^2 - bc + c^2\right) + \frac{a(b + c + 2a)(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(b^2 - bc + c^2\right) + \frac{a(2s - a + 2a)(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a).\frac{4b^2 + 4c^2 - 4bc + a(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a).$$

$$\frac{4(z + x)^2 + 4(x + y)^2 - 4(z + x)(x + y) + (y + z)\big((z + x) + (x + y) - 2(y + z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a = y + z, b = z + x, c = x + y)$$

$$= (2s + a).\frac{4x(x + y + z) + 2x(y + z) + 3(y - z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{3}{4}(b - c)^2 + \frac{a(s - a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{3}{4}(b - c)^2 + \frac{a(s - a)}{2}\right) - \frac{(a + 2s - 2s)(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{(b-c)^2}{2} + \frac{a(s - a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3 + c^3 - abc + a\left(4m_a^2\right) \overset{(\bullet\bullet)}{=} (2s + a)\left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2}\right) + \frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s - a) + (b - c)^2\left(\left(\frac{s}{2s+a}\right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s - a) - \frac{(b-c)^2}{4} + (b - c)^2.\left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$= s(s - a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s - a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s - a) + \frac{(b-c)^2}{4}\right)\left(s(s - a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s - a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}.\left(s(s - a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s - a)(b - c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq}0\rightarrow \text{true (strict inequality)} \therefore m_a n_a\geq p_a^2+\frac{(b-c)^2}{18}\rightarrow(3)$$

$$\text{Now, } (2m_a+n_a)^2-9p_a^2=4m_a^2+n_a^2+4m_an_a-9p_a^2\overset{\text{via }(3)}{\geq}$$

$$4s(s-a)+(b-c)^2+s(s-a)+\frac{s(b-c)^2}{a}+4p_a^2+\frac{2(b-c)^2}{9}-9p_a^2$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{=}5s(s-a)+\frac{11(b-c)^2}{9}+\frac{s(b-c)^2}{a}-5s(s-a)-\frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$=\left(\frac{11}{9}+\frac{s}{a}-\frac{5s(3s+a)}{(2s+a)^2}\right).(b-c)^2=\frac{36s^3-55s^2a+8sa^2+11a^3}{9a(2s+a)^2}.(b-c)^2$$

$$=\frac{(s-a)\left((s-a)(36s+17a)+6a^2\right)}{9a(2s+a)^2}.(b-c)^2\geq 0\ \therefore (2m_a+n_a)^2\geq 9p_a^2$$

$$\Rightarrow 2m_a+n_a\geq 3p_a \text{ and analogs}\rightarrow(4)$$

$$\text{Also, } p_a^2-m_a^2\overset{\text{via }(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\blacksquare)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\overset{s>a}{\geq}0\Rightarrow p_a\geq m_a\geq w_a\Rightarrow w_a\leq p_a$$

$$\therefore \text{ in order to prove : } \frac{p_a^2-m_a^2}{p_a+m_a} \le \frac{m_a^2-w_a^2}{m_a+w_a}, \text{ it suffices to prove : }$$

$$p_a^2-m_a^2 \le m_a^2-w_a^2$$

$$\overset{via\ (\blacksquare)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \le s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$=\frac{(b-c)^2}{4}\left(1+\frac{4s(s-a)}{(2s-a)^2}\right)=\frac{(b-c)^2}{4}.\frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$

$$\Leftrightarrow \left((2s-a)^2+4s(s-a)\right)(2s+a)^2 \ge (8s^2-a^2)(2s-a)^2$$

$$\Leftrightarrow 16s^3-12s^2a-4sa^2+a^3 \ge 0 \Leftrightarrow (s-a)(16s^2+4sa)+a^3 \ge 0$$

$$\to \text{true (strict) since } s > a$$

$$\therefore \frac{p_a^2-m_a^2}{p_a+m_a} \le \frac{m_a^2-w_a^2}{m_a+w_a} \Rightarrow \boxed{p_a+w_a \le 2m_a} \text{ and analogs} \to (5)$$

$$\text{Also, } p_a^2-m_a^2=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \overset{s>a}{\ge} 0 \Rightarrow p_a \ge m_a \Rightarrow p_a w_a \ge m_a w_a$$

$$\overset{Lascu+A-G}{\ge} s(s-a) \to (6)$$

$$\text{Now, } (3p_a+w_a)^2=9p_a^2+w_a^2+6p_a w_a \overset{via\ (\bullet\bullet\bullet)\ and\ (6)}{\ge}$$

$$9s(s-a)+\frac{9s(3s+a)(b-c)^2}{(2s+a)^2}+s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}+6s(s-a)$$

$$\overset{?}{\ge} 16m_a^2=16s(s-a)+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\ge} \frac{s(s-a)(b-c)^2}{(2s-a)^2}+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)}{(2s+a)^2} \overset{?}{\ge} \frac{s(s-a)+4(2s-a)^2}{(2s-a)^2} \quad (\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow 20t^4-36t^3+13t^2+5t-2 \overset{?}{\ge} 0 \quad \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\left((t-1)(20t^2+4t+1)+3\right) \overset{?}{\ge} 0 \to \text{true (strict)} \because t > 1$$

$$\therefore \boxed{3p_a+w_a \ge 4m_a} \text{ and analogs} \to (7)$$

$$\text{Finally, } an_a^2.ag_a^2 \ge a^2s^2(s-a)^2 \Leftrightarrow$$

$$\left(b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\right)\left(\begin{matrix}b^2(s-b)+c^2(s-c)\\-a(s-b)(s-c)\end{matrix}\right) \overset{(a)}{\ge} a^2s^2(s-a)^2$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$ and via such substitutions, $(a) \Leftrightarrow$

$$\left(z(z+x)^2+y(x+y)^2-yz(y+z)\right)\left(y(z+x)^2+z(x+y)^2-yz(y+z)\right)$$

$$\ge x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2+xz^2+y^3+z^3 \ge 2xyz+yz(y+z) \Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \ge 0$$

$$\to \text{true} \Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \overset{(b)}{\ge} s(s-a)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$ and $b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)$ and adding these two, we get :

$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$

$$=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) + a^2 - (b-c)^2$$
$$\Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2+g_a^2)$$
$$\Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 = (b-c)^2 + 2s(s-a)$$
$$\Rightarrow n_a^2+g_a^2+2n_ag_a \overset{\text{via (b)}}{\geq} (b-c)^2 + 4s(s-a) \Rightarrow (n_a+g_a)^2 \geq 4m_a^2$$
$$\Rightarrow \boxed{n_a+g_a \geq 2m_a} \text{ and analogs} \to (8)$$

Now, $w_a + 2p_a \leq 4m_a - w_a$ is equivalent to (5) and then :
$4m_a - w_a \leq 3p_a$ is equivalent to (7) and also,
$3p_a \leq 2m_a + n_a$ is equivalent to (4) and finally,
$2m_a + n_a \leq g_a + 2n_a$ is equivalent to (8)
$\therefore w_a + 2p_a \leq 4m_a - w_a \leq 3p_a \leq 2m_a + n_a \leq g_a + 2n_a \ \forall \Delta ABC$ (QED)

1779. In any non − right ΔABC, the following relationships hold :

$$① \ \frac{a^n}{b^n+c^n}(\sec B+\sec C) + \frac{b^n}{c^n+a^n}(\sec C+\sec A) + \frac{c^n}{a^n+b^n}(\sec A+\sec B) \geq 6$$
$$② \ \frac{h_a^n}{h_b^n+h_c^n}(\sec B+\sec C) + \frac{h_b^n}{h_c^n+h_a^n}(\sec C+\sec A) + \frac{h_c^n}{h_a^n+h_b^n}(\sec A+\sec B) \geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C > 0$, $(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB + B^2\right) - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB$$
$$= 6\sum_{cyc}AB + 2\sum_{cyc}A^2 - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \to (1)$$

$$\text{Now, } \forall x,y,z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy + z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2 + xyz\sum_{cyc}x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have : } \frac{x}{y+z}.(\sec B+\sec C) + \frac{y}{z+x}.(\sec C+\sec A) + \frac{z}{x+y}.(\sec A+\sec B)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)\ (A=\sec A, B=\sec B, C=\sec C)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3}.\sqrt{\sum_{cyc}\sec A\sec B}$$

$$=\sqrt{3(\sec A\sec B\sec C)\sum_{cyc}\cos A}=\sqrt{3\left(\frac{4R^2}{s^2-4R^2-4Rr-r^2}\right)\left(\frac{R+r}{R}\right)}$$

$$\overset{\text{Gerretsen}}{\geq}\sqrt{3\left(\frac{4R^2}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}\right)\left(\frac{R+r}{R}\right)}=\sqrt{\frac{6R(R+r)}{r^2}}$$

$$\overset{\text{Euler}}{\geq}\sqrt{\frac{6(2r)(3r)}{r^2}}=6\ \therefore\ \frac{x}{y+z}.(\sec B+\sec C)+$$

$\frac{y}{z+x}.(\sec C+\sec A)+\frac{z}{x+y}.(\sec A+\sec B)\geq 6$ and choosing $x=a^n, y=b^n,$ $z=c^n$ and $x=h_a^n, y=h_b^n, z=h_c^n$ respectively, we get :

$$① \frac{a^n}{b^n+c^n}(\sec B+\sec C)+\frac{b^n}{c^n+a^n}(\sec C+\sec A)+\frac{c^n}{a^n+b^n}(\sec A+\sec B)\geq 6$$

$$② \frac{h_a^n}{h_b^n+h_c^n}(\sec B+\sec C)+\frac{h_b^n}{h_c^n+h_a^n}(\sec C+\sec A)+\frac{h_c^n}{h_a^n+h_b^n}(\sec A+\sec B)\geq 6$$

$\forall$ non − right Δ ABC, " = " iff Δ ABC is equilateral (QED)

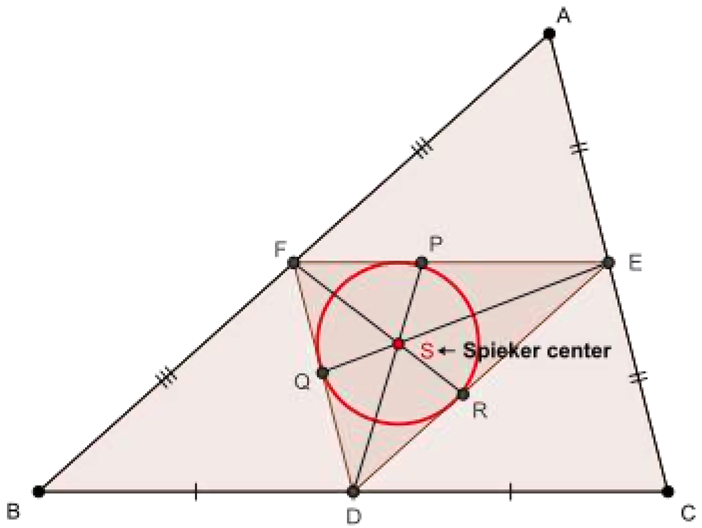
1780. **In any Δ ABC with**

$p_a\to$ Spieker cevian, the following relationship holds :

$$\frac{p_a}{h_a}\leq\frac{2R}{3r}-\frac{1}{3}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say)
and inr$a$dius of Δ DEF $= r'(say)$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $a$rrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$
$$\text{Again}, \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$
$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \stackrel{(****)}{=} \frac{r(a+c)}{2AS}$$
$$\text{Now}, [BAX]+[BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$
$$\stackrel{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$
$$\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$
$$\therefore \boxed{p_a^2 \stackrel{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$
$$\text{Now}, b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$
$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2)$$

$$= (2s + a)(b^2 - bc + c^2) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right)$$

$$= (2s + a)(b^2 - bc + c^2) + \frac{a(b + c + 2a)(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a)(b^2 - bc + c^2) + \frac{a(2s - a + 2a)(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a).\frac{4b^2 + 4c^2 - 4bc + a(b + c - 2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a).$$

$$\frac{4(z + x)^2 + 4(x + y)^2 - 4(z + x)(x + y) + (y + z)\big((z + x) + (x + y) - 2(y + z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a = y + z, b = z + x, c = x + y)$$

$$= (2s + a).\frac{4x(x + y + z) + 2x(y + z) + 3(y - z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{3}{4}(b - c)^2 + \frac{a(s - a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{3}{4}(b - c)^2 + \frac{a(s - a)}{2}\right) - \frac{(a + 2s - 2s)(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{(b-c)^2}{2} + \frac{a(s - a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3 + c^3 - abc + a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s + a)\left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2}\right) + \frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s - a) + (b - c)^2\left(\left(\frac{s}{2s+a}\right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s - a) - \frac{(b-c)^2}{4} + (b - c)^2.\left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$= s(s - a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s - a) + \frac{s(3s + a)(b - c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s - a) + \frac{(b-c)^2}{4}\right)\left(s(s - a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s - a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}.\left(s(s - a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s - a)(b - c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq}0\to\text{true (strict inequality)}\therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\to(3)$$

$$\text{Now, }(2m_a+n_a)^2-9p_a^2=4m_a^2+n_a^2+4m_an_a-9p_a^2\overset{\text{via (3)}}{\geq}$$

$$4s(s-a)+(b-c)^2+s(s-a)+\frac{s(b-c)^2}{a}+4p_a^2+\frac{2(b-c)^2}{9}-9p_a^2$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{=}5s(s-a)+\frac{11(b-c)^2}{9}+\frac{s(b-c)^2}{a}-5s(s-a)-\frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$=\left(\frac{11}{9}+\frac{s}{a}-\frac{5s(3s+a)}{(2s+a)^2}\right).(b-c)^2=\frac{36s^3-55s^2a+8sa^2+11a^3}{9a(2s+a)^2}.(b-c)^2$$

$$=\frac{(s-a)\left((s-a)(36s+17a)+6a^2\right)}{9a(2s+a)^2}.(b-c)^2\geq 0\therefore(2m_a+n_a)^2\geq 9p_a^2$$

$$\Rightarrow 2m_a+n_a\geq 3p_a\text{ and analogs}\to(4)$$

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=$$

$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a$$

$$\therefore a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\leq}4(R-r)^2s^2\Leftrightarrow$$

$$(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\leq}4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\geq}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\geq}0$$

$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\geq}0\to\text{true}\therefore an_a\leq 2Rs-2rs\Rightarrow\frac{n_a}{h_a}\leq\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$$

$$\Rightarrow \frac{n_a}{h_a} \le \frac{R}{r} - 1 \text{ and analogs} \rightarrow (5) \therefore (4) \Rightarrow \frac{3p_a}{h_a} \le \frac{2m_a}{h_a} + \frac{n_a}{h_a} \overset{\substack{\text{via } (5)\\ \text{and}\\ \text{Panaitopol}}}{\le} \frac{R}{r} + \frac{R}{r} - 1$$

$$\Rightarrow \frac{p_a}{h_a} \le \frac{2R}{3r} - \frac{1}{3}\ \forall\, \Delta\, ABC\ (QED)$$

1781.

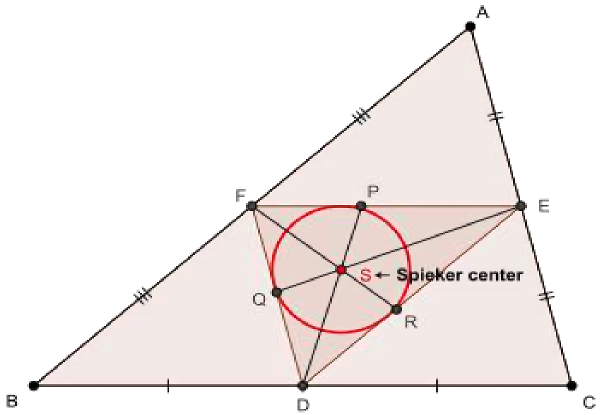
In any Δ ABC with:

$p_a, p_b, p_c \rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{p_a}{m_a} + \frac{p_b}{m_b} + \frac{p_c}{m_c} \le \frac{R}{2r} + 2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

Now, $\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

Again, $\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2) + a(b^2-bc+c^2) + a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2) + a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2) + a\left(\frac{(b+c)^2-(b-c)^2}{4} - a^2\right)$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a) + (b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 . \left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \stackrel{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \stackrel{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \stackrel{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \stackrel{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+ \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9} . \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \stackrel{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a) . \frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a) . \frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$

$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \stackrel{?}{\geq} 0 \left(\because (b-c)^2 \geq 0\right)$$

$$\Leftrightarrow \frac{s(s-a)\left(144s^3 - 52s^2a - 16sa^2 + 5a^3\right)}{36a(2s+a)^2}$$

$$+ \frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4} . (b-c)^2 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)\left(144s^2 + 92sa + 76a^2\right) + 81a^3\right)}{36a(2s+a)^2} +$$

$$\frac{(s-a)\left((s-a)\left(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3\right) + 1701a^4\right)}{324a(2s+a)^4} . (b-c)^2$$

$$\stackrel{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \rightarrow (3)$$

$$\text{Now, } (2m_a + n_a)^2 - 9p_a^2 = 4m_a^2 + n_a^2 + 4m_a n_a - 9p_a^2 \stackrel{\text{via }(3)}{\geq}$$

$$4s(s-a) + (b-c)^2 + s(s-a) + \frac{s(b-c)^2}{a} + 4p_a^2 + \frac{2(b-c)^2}{9} - 9p_a^2$$

$$\stackrel{\text{via }(\bullet\bullet\bullet)}{=} 5s(s-a) + \frac{11(b-c)^2}{9} + \frac{s(b-c)^2}{a} - 5s(s-a) - \frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$= \left(\frac{11}{9} + \frac{s}{a} - \frac{5s(3s+a)}{(2s+a)^2}\right) . (b-c)^2 = \frac{36s^3 - 55s^2a + 8sa^2 + 11a^3}{9a(2s+a)^2} . (b-c)^2$$

$$= \frac{(s-a)\left((s-a)(36s+17a) + 6a^2\right)}{9a(2s+a)^2} . (b-c)^2 \geq 0 \therefore (2m_a + n_a)^2 \geq 9p_a^2$$

$$\Rightarrow 2m_a + n_a \geq 3p_a \text{ and analogs} \therefore \frac{p_a}{m_a} + \frac{p_b}{m_b} + \frac{p_c}{m_c} \leq \sum_{\text{cyc}} \frac{2m_a + n_a}{3m_a}$$

$$= 2 + \sum_{cyc} \frac{n_a}{3m_a} \overset{?}{\le} \frac{R}{2r} + 2 \Leftrightarrow \boxed{\sum_{cyc} \frac{n_a}{m_a} \underset{(\blacksquare)}{\overset{?}{\le}} \frac{3R}{2r}}$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 =$$

$$as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \therefore n_a^2 = s^2 - 2r_ah_a$$

$$= as^2 - 2ah_ar_a \therefore n_a^2 = s^2 - 2r_ah_a = s^2 - \frac{4rs^2\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}} = s^2 - \frac{rs^2}{R}\sec^2\frac{A}{2}$$

$$\Rightarrow n_a^2(s-b)(s-c) = s^2(s-b)(s-c) - \frac{rs^2}{R}.\frac{(s-b)(s-c)}{s(s-a)}.bc$$

$$= s^2(s-b)(s-c) - \frac{rs^2}{R}.\tan^2\frac{A}{2}.\frac{4Rrs}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}$$

$$= s^2(s-b)(s-c) - \frac{r^2s^3}{R}.\tan\frac{A}{2}\left(1+\tan^2\frac{A}{2}\right) \therefore n_a^2(s-b)(s-c)$$

$$= s^2(s-b)(s-c) - \frac{r^2s^2}{R}.r_a - \frac{r^2}{R}.r_a^3 \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\left(n_a^2(s-b)(s-c)\right) = s^2\sum_{cyc}(s-b)(s-c) - \frac{r^2s^2}{R}.\sum_{cyc} r_a - \frac{r^2}{R}.\sum_{cyc} r_a^3$$

$$= s^2(4Rr + r^2) - \frac{r^2s^2(4R+r)}{R} - \frac{r^2}{R}.\left((4R+r)^3 - 3\prod_{cyc}\left(4R\cos^2\frac{A}{2}\right)\right)$$

$$= s^2(4Rr + r^2) - \frac{r^2s^2(4R+r)}{R} - \frac{r^2}{R}.\left((4R+r)^3 - 3.64R^3.\frac{s^2}{16R^2}\right)$$

$$= \frac{r\left((4R^2 + 9Rr - r^2)s^2 - r(4R+r)^3\right)}{R}$$

$$\Rightarrow \frac{\sum_{cyc}\left(n_a^2(s-b)(s-c)\right)}{s(s-a)(s-b)(s-c)} = \frac{r\left((4R^2 + 9Rr - r^2)s^2 - r(4R+r)^3\right)}{Rr^2s^2}$$

$$\Rightarrow \boxed{\sum_{cyc}\frac{n_a^2}{s(s-a)} = \frac{r\left((4R^2 + 9Rr - r^2)s^2 - r(4R+r)^3\right)}{Rr^2s^2}} \therefore \sum_{cyc}\frac{n_a}{m_a} \overset{\text{Lascu + A-G}}{\le}$$

$$\sum_{cyc}\frac{n_a}{\sqrt{s(s-a)}} \overset{\text{CBS}}{\le} \sqrt{3\sum_{cyc}\frac{n_a^2}{s(s-a)}} = \sqrt{\frac{3r\left((4R^2 + 9Rr - r^2)s^2 - r(4R+r)^3\right)}{Rr^2s^2}} \overset{?}{\le} \frac{3R}{2r}$$

$$\Leftrightarrow (3R^3 - 16R^2r - 36Rr^2 + 4r^3)s^2 + 4r^2(4R+r)^3 \underset{(\blacksquare\blacksquare)}{\overset{?}{\ge}} 0$$

$\boxed{\textit{Case 1}}$ $3R^3 - 16R^2r - 36Rr^2 + 4r^3 \ge 0$ *and then* : LHS of $(\blacksquare\blacksquare) \ge 4r^2(4R+r)^3$ $> 0 \Rightarrow (\blacksquare\blacksquare)$ is true (strict inequality)

$\boxed{Case\ 2}$ $3R^3-16R^2r-36Rr^2+4r^3<0$ and then : LHS of $(\blacksquare\blacksquare)\overset{\text{Gerretsen}}{\geq}$

$$(3R^3-16R^2r-36Rr^2+4r^3)(4R^2+4Rr+3r^2)+4r^2(4R+r)^3\overset{?}{\geq}0$$

$$\Leftrightarrow 12t^5-52t^4+57t^3+16t^2-44t+16\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left((t-2)(12t^2+20t+33)+70\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(\blacksquare\blacksquare)\text{ is true}$$

$$\therefore\text{ combining both cases, }(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true }\forall\,\Delta\,ABC\therefore\frac{p_a}{m_a}+\frac{p_b}{m_b}+\frac{p_c}{m_c}\leq\frac{R}{2r}+2$$

$\forall\,\Delta$ ABC,″ =″ iff Δ ABC is equilateral (QED)

1782. **In ΔABC the following relationship holds:**

$$\frac{n_a^2n_b^2}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{n_b^2n_c^2}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{n_c^2n_a^2}{\sin\frac{A}{2}\sin\frac{B}{2}}\geq 972r^4$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$h_a\leq n_a,h_b\leq n_b,h_c\leq n_c\text{ and }\sum\frac{1}{h_a}=\frac{1}{r},\sqrt[3]{h_ah_bh_c}\overset{GM\geq HM}{\geq}\frac{3}{\sum\frac{1}{h_a}}=3r(1)$$

$$\text{And }\sum\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{\left(\sum\sin\frac{B}{2}\right)^2}{3}\overset{Jensen}{\leq}\frac{\left(3\sin\frac{A+B+C}{6}\right)^2}{3}=\frac{\left(3.\frac{1}{2}\right)^2}{3}=\frac{3}{4}(2)$$

$$\frac{n_a^2n_b^2}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{n_b^2n_c^2}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{n_c^2n_a^2}{\sin\frac{A}{2}\sin\frac{B}{2}}\geq\sum\frac{h_a^2h_b^2}{\sin\frac{B}{2}\sin\frac{C}{2}}\overset{Bergstrom}{\geq}$$

$$\frac{(\sum h_ah_b)^2}{\sum\sin\frac{B}{2}\sin\frac{C}{2}}\overset{AM-GM\,\&(2)}{\geq}\frac{9(h_ah_bh_c)^{\frac{4}{3}}}{\frac{3}{4}}\overset{(1)}{\geq}12.(3r)^4=972r^4$$

Equality holds for $a=b=c$.

1783. **In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{a}{b\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}\right)+c\cos^2\frac{C}{2}}\geq\frac{4}{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum \cos^2\frac{A}{2} = 2 + \frac{r}{2R} \overset{Euler}{\leq} \frac{9}{4} (1)$$

$$\sum \frac{a}{b\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2}\right) + c\cos^2\frac{C}{2}} = \sum \frac{a^2}{ba\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2}\right) + ca\cos^2\frac{C}{2}} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(\sum a)^2}{(\sum ab)\left(\sum \cos^2\frac{A}{2}\right)} \overset{(1)}{\geq} \frac{(\sum a)^2}{\frac{(\sum a)^2}{3}\frac{9}{4}} = \frac{4}{3}$$

$$Equality\ for\ a = b = c$$

1784. **In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{1}{3\tan^2\frac{A}{2} + 2} \leq 1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{1}{3\tan^2\frac{A}{2} + 2} = \frac{1}{2}\sum\left(1 - \frac{3\tan^2\frac{A}{2}}{3\tan^2\frac{A}{2} + 2}\right) = \frac{3}{2} - \frac{1}{2}\sum\left(\frac{3\tan^2\frac{A}{2}}{3\tan^2\frac{A}{2} + 2}\right) \overset{CBS}{\leq}$$

$$\leq \frac{3}{2} - \frac{1}{2}.\frac{3\left(\sum \tan\frac{A}{2}\right)^2}{3\sum\tan^2\frac{A}{2} + 6} = \frac{1}{2}\left(3 - \frac{3\left(\frac{4R+r}{s}\right)^2}{3\left(\frac{(4R+r)}{s}\right)^2 - 6 + 6}\right) = \frac{1}{2}(3-1) = 1$$

$$Equality\ for\ \Delta ABC\ equilateral.$$

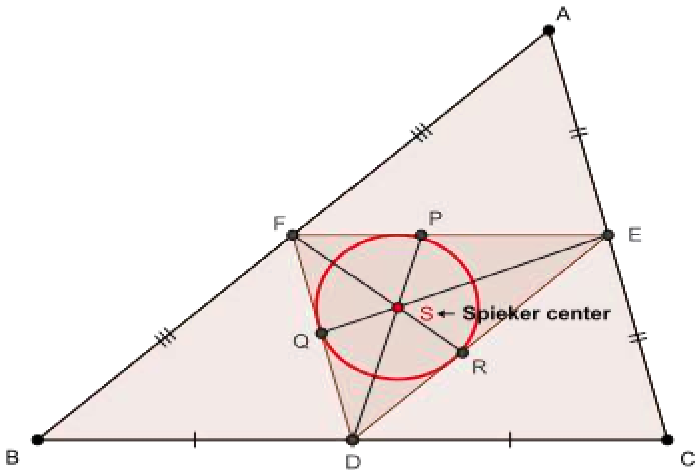
1785.

In any Δ ABC with
$p_a \to$ Spieker cevian, the following relationship holds :

$$\sum_{cyc} \frac{4p_a - h_a - r_a}{m_a} \leq \frac{m_a + m_b + m_c}{r}$$

Proposed by Bogdan Fuștei-Romania

Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say)
and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore\, m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=\mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2\mathrm{Rr}\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, }b^3+c^3-abc+a\left(4m_a^2\right)=b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$

$$=(b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$=2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a\le\frac{(b+c)^2}{16r}\Leftrightarrow p_a.\frac{2rs}{a}\le\frac{(b+c)^2}{8}.\frac{s}{a}\Leftrightarrow p_a h_a\le\frac{s(b+c)^2}{8a}\Leftrightarrow$$

$$p_a^2h_a^2\le\frac{s^2(b+c)^4}{64a^2}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)\le\frac{s^2(b+c)^4}{64a^2}$$

$$\Leftrightarrow s^2(s-a)^2-\frac{s^2(s-a)^2(b-c)^2}{a^2}+\frac{s(3s+a).s(s-a)(b-c)^2}{(2s+a)^2}$$

$$-\frac{s(3s+a).s(s-a)(b-c)^4}{a^2(2s+a)^2}\le\frac{s^2(b+c)^4}{64a^2}$$

$$\Leftrightarrow \frac{(2s-a)^4}{64a^2} - (s-a)^2 + (s-a)(b-c)^2\left(\frac{s-a}{a^2} - \frac{3s+a}{(2s+a)^2}\right)$$
$$+\frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \geq 0$$
$$\Leftrightarrow \frac{\left((2s-a)^2 - 8a(s-a)\right)\left((2s-a)^2 + 8a(s-a)\right)}{64a^2}$$
$$+\frac{(s-a)(b-c)^2\left((s-a)(2s+a)^2 - a^2(3s+a)\right)}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \geq 0$$
$$\Leftrightarrow \boxed{\frac{(2s-3a)^2(4s^2+4sa-7a^2)}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \overset{(\blacklozenge)}{\geq} 0}$$

$\boxed{Case\ 1}$ $(2s-3a)^2 \geq (b-c)^2$ *and then* : **LHS of** $(\blacklozenge) \geq$
$$\frac{(4s^2+4sa-7a^2)(b-c)^2}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} \overset{?}{\geq} 0$$
$$\Leftrightarrow \frac{4s^2+4sa-7a^2}{64} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)}{(2s+a)^2} \overset{?}{\geq} 0 \ (\because (b-c)^2 \geq 0)$$
$$\Leftrightarrow 272s^4 - 224s^3a - 392s^2a^2 + 232sa^3 + 121a^4 \boxed{\underset{(\blacksquare)}{\overset{?}{\geq}}} 0 \text{ and}$$

$\because 272(s-a)^4 + 864a(s-a)^3 > 0 \therefore$ **in order to prove** $(\blacksquare)$,
it suffices to prove : LHS of $(\blacksquare) > 272(s-a)^4 + 864a(s-a)^3$
$\Leftrightarrow 568t^2 - 1272t + 713 > 0 \ \left(t = \frac{s}{a}\right)$, **which is true** $\because$ **discriminant**
$= 1272^2 - 4(568)(713) = -1952 < 0 \Rightarrow (\blacksquare) \Rightarrow (\blacklozenge)$ **is true**

$\boxed{Case\ 2}$ $(b-c)^2 \geq (2s-3a)^2$ *and then* : **LHS of** $(\blacklozenge) \geq$
$$\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2}$$
$$+\frac{(3s+a)(s-a)(b-c)^2(2s-3a)^2}{a^2(2s+a)^2} = \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2}$$
$$+\frac{(s-a)(b-c)^2}{a^2(2s+a)^2}.\left(2(s+a)(2s^2-2sa-a^2) + (3s+a)(2s-3a)^2\right)$$
$$= \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2}.(16s^2-16sa-7a^2)$$

which is definitely ≥ 0 **if** : $16s^2 - 16sa - 7a^2 \geq 0$ *and* **so, we now consider** :
$16s^2 - 16sa - 7a^2 < 0$ *and* $\because (b-c)^2 < a^2 \therefore$ **LHS of** $(\blacklozenge) \geq$
$$\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2}.(16s^2-16sa-7a^2)$$
$$> \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(16s^2-16sa-7a^2)}{(2s+a)^2} \overset{?}{>} 0 \Leftrightarrow$$
$$64s^6 - 64s^5a + 752s^4a^2 - 2784s^3a^3 + 2812s^2a^4 - 260sa^5 - 511a^6 \boxed{\underset{(\blacksquare\blacksquare)}{\overset{?}{>}}} 0 \text{ and}$$

$$\because (4s-5a)^6+104a(s-a)(4s-5a)^4+437a^3(4s-5a)^4>0$$
$$\therefore \text{ in order to prove } (\blacksquare\blacksquare), \text{it suffices to prove :}$$
$$64\bullet\text{LHS of } (\blacksquare\blacksquare) > (4s-5a)^6+104a(s-a)(4s-5a)^4+437a^3(4s-5a)^4$$
$$\Leftrightarrow 79232t^3-280616t^2+329680t-128227>0$$
$$\Leftrightarrow (t-1)(79232t^2-201384t+128296)+69>0 \to \text{true} \because t=\frac{s}{a}>1 \text{ and}$$
$$\because \text{discriminant of } (79232t^2-201384t+128296)$$
$$=201384^2-4(79232)(128296)=-105079232$$
$$\Rightarrow 79232t^2-201384t+128296>0 \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacklozenge) \text{ is true}$$
$$\therefore \text{combining both cases, } (\blacklozenge) \text{ is true } \forall\, \Delta\, ABC \therefore p_a \le \frac{(b+c)^2}{16r}$$
$$\forall\, \Delta\, ABC, ''='' \text{ iff } 2s-3a=0 \text{ and } b=c \Rightarrow ''='' \text{ iff } \Delta\, ABC \text{ is equilateral} \to (3)$$
$$\text{Now, } 4rp_a-rh_a-rr_a \overset{\text{via (3)}}{\le} \frac{(b+c)^2}{4}-r\left(\frac{2rs}{a}+\frac{rs}{s-a}\right)$$
$$=\frac{(b+c)^2}{4}-\frac{r^2s(b+c-a+a)}{a(s-a)}=\frac{(b+c)^2}{4}-\frac{4(s-b)(s-c)(b+c)}{4a}$$
$$=\frac{(b+c)^2}{4}-\frac{(b+c)(a^2-(b-c)^2)}{4a}=\frac{a(b+c)^2-a^2(b+c)+(b+c)(b-c)^2}{4a}$$
$$\overset{?}{<} m_a^2 \Leftrightarrow a(b+c)^2-a^2(b+c)+(b+c)(b-c)^2 \overset{?}{<} a((b+c)^2-a^2+(b-c)^2)$$
$$\Leftrightarrow a^2(b+c-a) \overset{?}{>} (b+c-a)(b-c)^2 \Leftrightarrow 2(s-a)(a^2-(b-c)^2) \overset{?}{>} 0$$
$$\Leftrightarrow 8(s-a)(s-b)(s-c) \overset{?}{>} 0 \to \text{true} \therefore 4rp_a-rh_a-rr_a<m_a^2$$
$$\Rightarrow \frac{4p_a-h_a-r_a}{m_a}<\frac{m_a}{r} \text{ and analogs} \therefore \sum_{cyc}\frac{4p_a-h_a-r_a}{m_a}<\frac{m_a+m_b+m_c}{r}$$
$$\forall\, \Delta\, ABC \text{ (QED)}$$

1786.

If $a=\min\{a,b,c\}$, then in acute Δ ABC, the following relationship holds :

$$2\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right) \ge \frac{R}{r}+\frac{h_b+h_c}{h_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{b+c}{a} \overset{?}{\ge} \frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$
$$\Leftrightarrow (b+c)(a+b-c).(b+c-a)(c+a-b) \overset{?}{\ge} 2a^2bc$$
$$\Leftrightarrow (b+c)(a+b-c).(b+c-a)(c+a-b) \overset{?}{\ge} 2a^2bc$$
$$\Leftrightarrow (ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)$$
$$\overset{?}{\ge} 2a^2bc$$

$$\Leftrightarrow 2a^2b^2 + 2a^2bc + 2ab(b^2 - c^2) - (a^2 + b^2 - c^2)(ab + ac + b^2 - c^2) \overset{?}{\geq} 2a^2bc$$

$$\Leftrightarrow 2a^2b^2 - (a^2 + b^2 - c^2)(ab + ac) + 2ab(b^2 - c^2) - 2ab(b^2 - c^2).\cos C \overset{?}{\geq} 0$$

$$\Leftrightarrow 2a^2b^2 - a^2(ab + ac) - (b^2 - c^2)(ab + ac) + 2ab(b^2 - c^2).2\sin^2\frac{C}{2} \overset{?}{\geq} 0$$

$$\Leftrightarrow a^2(2b^2 - ab - ac) - (b^2 - c^2)(ab + ac) + (b^2 - c^2)(c^2 - (a - b)^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow a^2(2b^2 - ab - ac) + (b^2 - c^2)(c^2 - a^2 - b^2 + 2ab - ab - ac) \overset{?}{\geq} 0$$

$$\Leftrightarrow \left((a^2 - b^2 + c^2) + (b^2 - c^2)\right)(2b^2 - ab - ac)$$

$$+(b^2 - c^2)(c^2 - a^2 - b^2 + ab - ac) \overset{?}{\geq} 0$$

$$\Leftrightarrow (c^2 + a^2 - b^2)(2b^2 - ab - ac) + (b^2 - c^2)(b^2 + c^2 - a^2 - 2ac) \overset{?}{\geq} 0$$

$$\Leftrightarrow (b^2 - c^2)(b^2 + c^2 - 2ac) - (b^2 - c^2)(2b^2 - ab - ac) + a^2(2b^2 - ab - ac)$$

$$-a^2(b^2 - c^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow (b^2 - c^2)\left((c^2 - ca) - (b^2 - ab)\right) + a^2\left((c^2 - ca) + (b^2 - ab)\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow (c^2 - ca)(b^2 - c^2 + a^2) + (b^2 - ab)(a^2 + c^2 - b^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow c(c - a)(a^2 + b^2 - c^2) + b(b - a)(c^2 + a^2 - b^2) \overset{?}{\geq} 0 \to \text{true}$$

$\because \Delta\, ABC$ being acute $\Rightarrow (a^2 + b^2 - c^2), (c^2 + a^2 - b^2) > 0$ and $a = \min\{a, b, c\}$

$$\Rightarrow (c - a), (b - a) \geq 0 \therefore \frac{R}{r} + \frac{h_b + h_c}{h_a} \leq \frac{b + c}{a} + \frac{ca + ab}{bc} = \frac{b}{a} + \frac{a}{b} + \frac{c}{a} + \frac{a}{c}$$

$$= \frac{c^2 + a^2}{ca} + \frac{a^2 + b^2}{ab} \overset{\text{Tereshin}}{\leq} \frac{4Rm_b}{2Rh_b} + \frac{4Rm_c}{2Rh_c} = 2\left(\frac{m_b}{h_b} + \frac{m_c}{h_c}\right)$$

$$\therefore 2\left(\frac{m_b}{h_b} + \frac{m_c}{h_c}\right) \geq \frac{R}{r} + \frac{h_b + h_c}{h_a} \text{ in acute } \Delta\, ABC \text{ with } a = \min\{a, b, c\} \text{ (QED)}$$

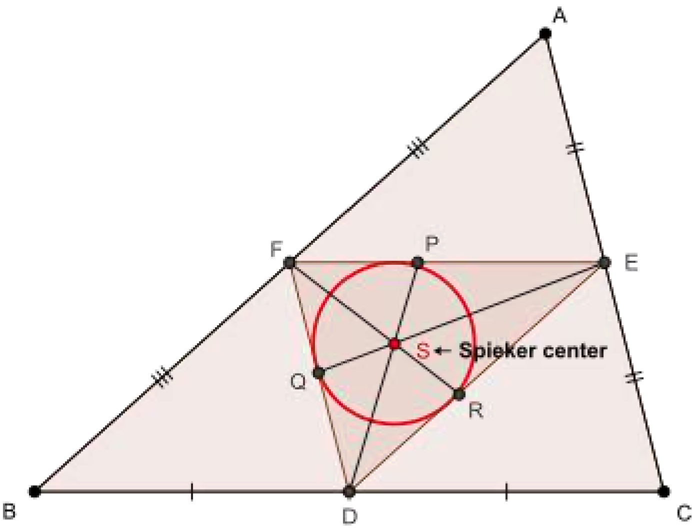
1787.

In any $\Delta\, ABC$ with p_a, p_b, p_c
$\to$ Spieker cevians, the following relationship holds :

$$1 + \sqrt[3]{\prod_{cyc} \frac{n_a + p_a + w_a}{h_a}} \leq \frac{2R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



$$\textbf{Let AS produced meet BC } at\ \mathbf{X}\ and\ \mathbf{m}(\measuredangle \mathbf{BAX}) = \alpha\ and\ \mathbf{m}(\measuredangle \mathbf{CAX}) = \beta\ (say)$$

$$and\ \textbf{inradius of } \Delta\,\mathbf{DEF} = \mathbf{r}'\,(say)$$

$$\textbf{Now, } 16[\mathrm{DEF}]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [\mathrm{DEF}] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \textbf{ Spieker center is incenter of } \Delta\,\mathrm{DEF}, \therefore\ m(\measuredangle \mathrm{AFS}) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2}\ and\ m(\measuredangle \mathrm{AES}) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$\textbf{Via } (1),(2)\ and\ \textbf{using cosine law on } \Delta\,\mathrm{AFS}\ and\ \Delta\,\mathrm{AES}, \textbf{ we arrive at}:$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\textbf{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2\mathrm{Rr}$$

$$\text{Again,} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Also, } p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right) - m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$
$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$
$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$
$$=\frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$
$$=\frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$
$$\therefore p_a^2-m_a^2\overset{(\blacksquare\blacksquare)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\overset{s>a}{\geq}0\Rightarrow p_a\geq m_a\geq w_a\Rightarrow w_a\leq p_a$$
$$\therefore \text{in order to prove : }\frac{p_a^2-m_a^2}{p_a+m_a}\leq\frac{m_a^2-w_a^2}{m_a+w_a},\text{it suffices to prove : }$$
$$p_a^2-m_a^2\leq m_a^2-w_a^2$$
$$\overset{via\,(\blacksquare\blacksquare)}{\Leftrightarrow}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\leq s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$
$$=\frac{(b-c)^2}{4}\left(1+\frac{4s(s-a)}{(2s-a)^2}\right)=\frac{(b-c)^2}{4}\cdot\frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$
$$\Leftrightarrow\left((2s-a)^2+4s(s-a)\right)(2s+a)^2\geq(8s^2-a^2)(2s-a)^2$$
$$\Leftrightarrow 16s^3-12s^2a-4sa^2+a^3\geq 0\Leftrightarrow(s-a)(16s^2+4sa)+a^3\geq 0\to\text{true}$$
$$(\text{strict})\text{ since }s>a\therefore\frac{p_a^2-m_a^2}{p_a+m_a}\leq\frac{m_a^2-w_a^2}{m_a+w_a}\Rightarrow p_a+w_a\leq 2m_a\text{ and analogs}\to(3)$$
$$\text{Again, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2$$
$$=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a$$
$$\therefore a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\leq}4(R-r)^2s^2$$
$$\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\leq}4(R^2-2Rr+r^2)s^2$$
$$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\geq}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\geq}0$$
$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\geq}0\to\text{true}\therefore an_a\leq 2Rs-2rs\Rightarrow\frac{n_a}{h_a}\leq\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$$
$$\Rightarrow\frac{n_a}{h_a}\leq\frac{R}{r}-1\text{ and analogs}\to(4)\therefore(3)\text{ and }(4)\Rightarrow\frac{n_a+p_a+w_a}{h_a}\leq$$
$$\frac{R}{r}-1+\frac{2m_a}{h_a}\overset{Panaitopol}{\leq}\frac{R}{r}-1+\frac{R}{r}\therefore\frac{n_a+p_a+w_a}{h_a}\leq\frac{2R}{r}-1\text{ and analogs}$$

$$\Rightarrow 1+\sqrt[3]{\prod_{cyc}\frac{n_a+p_a+w_a}{h_a}}\le 1+\sqrt[3]{\left(\frac{2R}{r}-1\right)^3}=\frac{2R}{r}$$

$$\therefore 1+\sqrt[3]{\prod_{cyc}\frac{n_a+p_a+w_a}{h_a}}\le\frac{2R}{r}\ \forall\,\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1788. In ΔABC the following relationship holds:

$$10-\frac{2r}{R}\le\sum a\sum\frac{1}{a}\le\frac{9R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum a\sum\frac{1}{a}=2s\frac{s^2+r^2+4Rr}{4Rrs}=\frac{s^2+r^2+4Rr}{2Rr}$$

$$\sum a\sum\frac{1}{a}=\frac{s^2+r^2+4Rr}{2Rr}\overset{Gerretsen}{\le}\frac{4R^2+8Rr+4r^2}{2Rr}=$$

$$=\frac{4(R+r)^2}{2Rr}\overset{Euler}{\le}\frac{4\left(R+\frac{R}{2}\right)^2}{2Rr}=\frac{9R}{2r}$$

$$\sum a\sum\frac{1}{a}=\frac{s^2+r^2+4Rr}{2Rr}\overset{Gerretsen}{\ge}\frac{20Rr-4r^2}{2Rr}=10-\frac{4r}{2R}=10-\frac{2r}{R}$$

Equality holds for $a=b=c$.

1789. In ΔABC, $m(BAC)=90°, AD\perp BC$, R_1,R_2-circumradii and r_1,r_2-inradii

Prove that:

$$\frac{r_a+r_b+r_c}{r+r_1+r_2}\ge\left(\sqrt{2}+1\right)\frac{r_a+r_b+r_c}{R+R_1+R_2}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

Obviously,we have to prove that:

$$\frac{R+R_1+R_2}{r+r_1+r_2}\ge\sqrt{2}+1\quad(1)$$

It is known that:

$$\text{In } \Delta ABD\ (\measuredangle D=90^\circ)\ \ R_1=\frac{c}{2},r_1=\frac{m+h-c}{2}$$
$$\text{In } \Delta ACD\ (\measuredangle D=90^\circ)\ \ R_2=\frac{b}{2},r_2=\frac{m+h-b}{2}$$
$$\text{In } \Delta BAC\ (\measuredangle A=90^\circ)\ \ R=\frac{a}{2},r=\frac{b+c-a}{2}$$

Consider the given (1)*:*

$$\frac{R+R_1+R_2}{r+r_1+r_2}=\frac{\frac{a}{2}+\frac{c}{2}+\frac{b}{2}}{\frac{b+c-a}{2}+\frac{m+h-c}{2}+\frac{n+h-b}{2}}=$$
$$=\frac{a+b+c}{b+c-a+m+h-c+n+h-b}=\frac{a+b+c}{\underbrace{(m+n)}_{a}+2h-a}=$$
$$=\frac{a+b+c}{2h}=\frac{2p}{2h}=\frac{p}{h}=\frac{\frac{S}{r}}{\frac{2S}{a}}=\frac{a}{2r}$$
$$\frac{R+R_1+R_2}{r+r_1+r_2}=\frac{a}{2r}=\frac{a}{b+c-a}=\frac{2R}{2RsinB+2RsinC-2R}=\frac{2R}{2R(sinB+sinC-1)}=$$
$$=\frac{1}{sinB+sinC-1}=\frac{1}{2sin\frac{B+C}{2}\cdot cos\frac{B-C}{2}-1}=\frac{1}{\sqrt{2}cos\frac{B-C}{2}-1}$$
$$sin\frac{B+C}{2}=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}\ ;\ 0\le\cos\frac{B-C}{2}\le 1$$

Therefore

$$\frac{R+R_1+R_2}{r+r_1+r_2}=\frac{1}{\sqrt{2}cos\frac{B-C}{2}-1}\ge\frac{1}{\sqrt{2}-1}=\sqrt{2}+1\ \ (qed)$$

So,

$$\frac{R+R_1+R_2}{r+r_1+r_2}\ge\sqrt{2}+1$$

1790. In ΔABC the following relationship holds:

$$\sum_{cyc}\tan\frac{A}{2}\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\ge 1$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

Note In any ΔABC:

$$\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1\text{ , Let }\tan\frac{A}{2} = x, \tan\frac{B}{2} = y, \tan\frac{C}{2} = z$$

$$\text{Now }\sum xy = 1\text{ we will show:}$$

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq 1$$

Proof:

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq 1$$

$$\text{or }(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq (xy+yz+zx)^3\ \left(as\ \sum xy = 1\right)$$

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) =$$

$$= (xy+x^2+y^2)(y^2+z^2+yz)(x^2+zx+z^2) \overset{Holder}{\geq}$$

$$\geq (xy+yz+zx)^3 = 1$$

$$WLOG\ a \geq b \geq c\ so\ \tan\frac{A}{2} \geq \tan\frac{B}{2} \geq, \tan\frac{C}{2}$$

$$\sum \tan\frac{A}{2}\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)} \overset{Chebyshev}{\geq}$$

$$\geq \frac{1}{3}\sum \tan\frac{A}{2}\ \sum\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)} \overset{AM-GM}{\geq}$$

$$\frac{1}{3}\cdot\frac{1}{\sqrt{3}}\frac{4R+r}{s}.3\left(\prod\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\right)^{\frac{1}{3}} \overset{\frac{4R+r}{s}\geq\sqrt{3}}{\geq}$$

$$\geq \frac{1}{3}\cdot\frac{1}{\sqrt{3}}.\sqrt{3}.(1)^{\frac{1}{3}} = 1$$

Equality holds for $a = b = c$.

1791. **In ΔABC the following relationship holds:**

$$27\left(\sum_{cyc} a^2\right)^2 - 54\sum_{cyc} a^4 \leq 16s^4$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$16F^2 = 2\sum a^2b^2 - \left(\sum a^4\right)$$

$$27\left(\sum a^2\right)^2-54\sum a^4=27\sum a^4+54\sum a^2b^2-54\sum a^4=$$

$$=27\cdot\left(2\sum a^2b^2-\left(\sum a^4\right)\right)=27\cdot(16F^2)=$$

$$=27\cdot 16\cdot r^2s^2\overset{Mitrinovic}{\leq}27\cdot 16\cdot\frac{s^2}{27}\cdot s^2=16s^4$$

Equality holds for $a=b=c$.

1792. In ΔABC the following relationship holds:

$$\frac{m_a}{m_b+m_c}\cot\frac{A}{2}+\frac{m_b}{m_c+m_a}\cot\frac{B}{2}+\frac{m_c}{m_a+m_b}\cot\frac{C}{2}\geq\frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c$$

$$then\ m_a\leq m_b\leq m_c,\ m_a+m_b\leq m_a+m_c\leq m_b+m_c,$$
$$\cot\frac{A}{2}\leq\cot\frac{B}{2}\leq\cot\frac{C}{2}$$

$$\frac{m_a}{m_b+m_c}\cot\frac{A}{2}+\frac{m_b}{m_c+m_a}\cot\frac{B}{2}+\frac{m_c}{m_a+m_b}\cot\frac{C}{2}\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\sum\frac{m_a}{m_b+m_c}\sum\cot\frac{A}{2}\overset{Nesbitt}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{3}{2}\cdot\frac{s}{r}\overset{Mitrinovic}{\geq}\frac{1}{2}\cdot\frac{3\sqrt{3}r}{r}=\frac{3\sqrt{3}}{2}$$

$$Equality\ for\Delta ABC\ equilateral$$

1793. In ΔABC the following relationship holds:

$$3(ab+bc+ca)\geq a^2+b^2+c^2+36Rr$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$ab + bc + ca = s^2 + r^2 + 4Rr$$

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$$

We must prove that:

$$3(s^2 + r^2 + 4Rr) \geq 2(s^2 - r^2 - 4Rr) + 36Rr$$

$$3s^2 + 3r^2 + 12Rr \geq 2s^2 - 2r^2 - 8Rr + 36Rr$$

$$s^2 \geq 16Rr - 5r^2$$

which is Gerretsen's inequality.

Equality holds for $a = b = c$.

1794. If $n \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$\frac{a^n}{sinA} + \frac{b^n}{sinB} + \frac{c^n}{sinC} \geq \left(2\sqrt{3}\right)^{n+1} \cdot r^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{a^n}{sinA} + \frac{b^n}{sinB} + \frac{c^n}{sinC} = \sum_{cyc} \frac{a^n}{sinA} =$$

$$= \sum_{cyc} \frac{a^n}{\frac{a}{2R}} = 2R \cdot \sum_{cyc} a^{n-1} = 2R \cdot \sum_{cyc} \frac{a^{n-1}}{1^{n-2}} \overset{RADON}{\geq}$$

$$\geq 2R \cdot \frac{(a+b+c)^{n-1}}{(1+1+1)^{n-2}} = 2R \cdot \frac{(2s)^{n-1}}{(3)^{n-2}} \overset{MITRINOVIC}{\geq}$$

$$\geq 2R \cdot \frac{2^{n-1} \cdot \left(3\sqrt{3}r\right)^{n-1}}{3^{n-2}} \overset{EULER}{\geq} 2 \cdot 2r \cdot 2^{n-1} \cdot 3^{\frac{3n-3}{2} - n + 2} \cdot r^{n-1} =$$

$$= 2^{n+1} \cdot 3^{\frac{n+1}{2}} \cdot r^n = \left(2\sqrt{3}\right)^{n+1} \cdot r^n$$

Equality holds for $a = b = c$.

1795. If $n \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$\frac{h_a^n}{sinA} + \frac{h_b^n}{sinB} + \frac{h_c^n}{sinC} \geq 2 \cdot 3^{\frac{2n+1}{2}} \cdot r^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{h_a^n}{sinA} + \frac{h_b^n}{sinB} + \frac{h_c^n}{sinC} = \sum_{cyc} \frac{h_a^n}{sinA} = \sum_{cyc} \frac{\left(\frac{2F}{a}\right)^n}{sinA} =$$

$$= (2F)^n \cdot \sum_{cyc} \frac{1}{a^n sinA} = (2rs)^n \cdot \sum_{cyc} \frac{1}{a^n \cdot \frac{a}{2R}} =$$

$$= 2^{n+1} \cdot r^n \cdot s^n \cdot R \sum_{cyc} \frac{1^{n+2}}{a^{n+1}} \overset{RADON}{\geq} 2^{n+1} \cdot r^n \cdot s^n \cdot R \cdot \frac{(1+1+1)^{n+2}}{(a+b+c)^{n+1}} =$$

$$= 2^{n+1} \cdot r^n \cdot s^n \cdot R \cdot \frac{(3)^{n+2}}{(2s)^{n+1}} = r^n \cdot R \cdot \frac{3^{n+2}}{s} \overset{MITRINOVIC}{\geq}$$

$$\geq r^n \cdot R \cdot \frac{3^{n+2}}{\frac{3\sqrt{3}}{2} \cdot R} = 2 \cdot 3^{n+2-\frac{3}{2}} \cdot r^n = 2 \cdot 3^{\frac{2n+1}{2}} \cdot r^n$$

Equality holds for $a = b = c$.

1796. In ΔABC the following relationship holds:

$$\frac{n_a^4}{sinA} + \frac{n_b^4}{sinB} + \frac{n_c^4}{sinC} \geq 162\sqrt{3}r^4$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{n_a^4}{sinA}+\frac{n_b^4}{sinB}+\frac{n_c^4}{sinC}=\sum_{cyc}\frac{n_a^4}{sinA}\ge\sum_{cyc}\frac{h_a^4}{sinA}=$$

$$=\sum_{cyc}\frac{\left(\frac{2F}{a}\right)^4}{sinA}=16F^4\sum_{cyc}\frac{1}{a^4 sinA}=16F^4\sum_{cyc}\frac{1}{a^4\cdot\frac{a}{2R}}=$$

$$=32F^4R\sum_{cyc}\frac{1}{a^5}=32r^4s^4R\sum_{cyc}\frac{1^6}{a^5}\overset{RADON}{\ge}$$

$$\ge\ 32r^4s^4R\cdot\frac{(1+1+1)^6}{(a+b+c)^5}=\frac{32r^4s^4R\cdot 3^6}{32s^5}=$$

$$=\frac{3^6r^4R}{s}\overset{MITRINOVIC}{\ge}\frac{3^6r^4R}{\frac{3\sqrt{3}}{2}R}=\frac{2\cdot 243}{\sqrt{3}}r^4=162\sqrt{3}r^4$$

Equality holds for: $a=b=c$.

1797. In ΔABC the following relationship holds:

$$\frac{h_a^3}{sinA}+\frac{h_b^3}{sinB}+\frac{h_c^3}{sinC}\ge\ 54\sqrt{3}r^3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{h_a^3}{sinA}+\frac{h_b^3}{sinB}+\frac{h_c^3}{sinC}=\sum_{cyc}\frac{h_a^3}{sinA}=\sum_{cyc}\frac{\left(\frac{2F}{a}\right)^3}{sinA}=8F^3\sum_{cyc}\frac{1}{a^3 sinA}=$$

$$=\ 8F^3\sum_{cyc}\frac{1}{a^3\cdot\frac{a}{2R}}=16F^3R\sum_{cyc}\frac{1}{a^4}=16r^3s^3R\sum_{cyc}\frac{1^5}{a^4}\overset{RADON}{\ge}$$

$$\ge\ 16r^3s^3R\cdot\frac{(1+1+1)^5}{(a+b+c)^4}=16r^3s^3R\cdot\frac{3^5}{(2s)^4}=\frac{r^3s^3R\cdot 243}{s^4}=$$

$$= \frac{r^3R\cdot 243}{s} \overset{MITRINOVIC}{\geq} \frac{r^3R\cdot 243}{\frac{3\sqrt{3}}{2}\cdot R} = \frac{81r^3}{\sqrt{3}}\cdot 2 = 54\sqrt{3}r^3$$

Equality holds for $a = b = c$.

1798. **In any Δ ABC with**

$n_a \to$ Nagel cevian, the following relationship holds :

$$n_a \geq \frac{b^2 - bc + c^2}{2R}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - s(a^2 - (b-c)^2) = as(s-a) + s(b-c)^2$$
$$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2 \to (1)$$

$$\text{Now, } n_a \geq \frac{b^2-bc+c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \geq \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \geq \left(\frac{b^2-bc+c^2}{bc}\right)^2 - 1$$
$$= \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \Leftrightarrow \frac{n_a^2 - h_a^2}{h_a^2} \geq \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \overset{\text{via (1)}}{\Leftrightarrow}$$
$$s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2} \geq \frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow \left(\frac{s}{a} + \frac{s(s-a)}{a^2}\right)(b-c)^2 \geq \frac{(b-c)^2(b^2+c^2)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \geq \frac{b^2+c^2}{4R^2}\ (\because (b-c)^2 \geq 0)$$
$$\Leftrightarrow 4R^2s^2 \geq a^2b^2 + c^2a^2 \to \text{true (strict inequality)} \because 4R^2s^2 \overset{\text{Goldstone}}{\geq}$$
$$\sum_{\text{cyc}} a^2b^2 > a^2b^2 + c^2a^2 \therefore n_a \geq \frac{b^2-bc+c^2}{2R}, \text{"} = \text{" iff } b = c \text{ (QED)}$$

1799.

In any Δ ABC with n_a, n_b, n_c

$\to$ Nagel's cevians, the following relationship holds :

$$\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \geq \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} - 3$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco
Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a)+bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - s(a^2-(b-c)^2) = as(s-a) + s(b-c)^2$$
$$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2 \rightarrow (1)$$
$$\text{Now}, \frac{n_a}{h_a} \overset{?}{\geq} \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \overset{?}{\geq} \left(\frac{b^2-bc+c^2}{bc}\right)^2 - 1 = \frac{(b-c)^2(b^2+c^2)}{b^2c^2}$$
$$\Leftrightarrow \frac{n_a^2 - h_a^2}{h_a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \overset{\text{via (1)}}{\Leftrightarrow}$$
$$s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow \left(\frac{s}{a} + \frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \overset{?}{\geq} \frac{b^2+c^2}{4R^2} \ (\because (b-c)^2 \geq 0)$$
$$\Leftrightarrow 4R^2s^2 \overset{?}{\geq} a^2b^2 + c^2a^2 \rightarrow \text{true (strict inequality)} \because 4R^2s^2 \overset{\text{Goldstone}}{\geq}$$
$$\sum_{\text{cyc}} a^2b^2 > a^2b^2 + c^2a^2 \therefore \frac{n_a}{h_a} \geq \frac{b^2-bc+c^2}{bc} \text{ and analogs}$$
$$\Rightarrow \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \geq \sum_{\text{cyc}} \frac{b^2-bc+c^2}{bc} = \sum_{\text{cyc}}\left(\frac{b}{c}+\frac{c}{b}\right) - 3 = \sum_{\text{cyc}} \frac{b+c}{a} - 3$$
$$\therefore \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \geq \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} - 3$$
$$\forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1800. In any Δ ABC with $n_a, n_b, n_c \rightarrow$ Nagel's cevians, the following relationship holds :

$$\frac{\sqrt{n_bn_c}}{h_a} + \frac{\sqrt{n_cn_a}}{h_b} + \frac{\sqrt{n_an_b}}{h_c} \geq \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s(a^2-(b-c)^2)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2\to(1)$$
$$\text{Now, } n_a\overset{?}{\geq}\frac{b^2-bc+c^2}{2R}\Leftrightarrow\frac{n_a}{h_a}\overset{?}{\geq}\frac{b^2-bc+c^2}{bc}\Leftrightarrow\frac{n_a^2}{h_a^2}-1\overset{?}{\geq}\left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$
$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\geq}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\overset{\text{via }(1)}{\Leftrightarrow}$$
$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}\overset{?}{\geq}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}.\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\geq}\frac{(b-c)^2(b^2+c^2)}{4R^2}\Leftrightarrow\frac{s^2}{a^2}\overset{?}{\geq}\frac{b^2+c^2}{4R^2}\ (\because(b-c)^2\geq 0)$$
$$\Leftrightarrow 4R^2s^2\overset{?}{\geq}a^2b^2+c^2a^2\to\text{true (strict inequality)}\because 4R^2s^2\overset{\text{Goldstone}}{\geq}\sum_{cyc}a^2b^2>$$
$$a^2b^2+c^2a^2\therefore n_a\overset{(\bullet)}{\geq}\frac{b^2-bc+c^2}{2R}\text{ and analogs}$$
$$\therefore 4R^2.n_bn_c\geq(c^2-ca+a^2)(a^2-ab+b^2)$$
$$=((x+y)^2-(x+y)(y+z)+(y+z)^2)((y+z)^2-(y+z)(z+x)+(z+x)^2)$$
$$(x=s-a,y=s-b,z=s-c)=\sum_{cyc}x^4+\sum_{cyc}x^2y^2+2yz(y^2+z^2+yz)+4x^2yz$$
$$=(y^4+z^4+2y^2z^2)+(x^4+y^2z^2+2x^2yz)+(x^2y^2+x^2z^2+2x^2yz)$$
$$+2yz(y^2+z^2)\geq(y^2+z^2)^2+(x^2+yz)^2+x^2(y+z)^2+yz(y+z)^2$$
$$\geq\frac{(y+z)^4}{4}+(x^2+yz)^2+(x^2+yz)(y+z)^2$$
$$=\frac{(y+z)^4+4(x^2+yz)^2+4(x^2+yz)(y+z)^2}{4}=\frac{\left((y+z)^2+2(x^2+yz)\right)^2}{4}$$
$$=\frac{\left(a^2+2\left((s-a)^2+(s-b)(s-c)\right)\right)^2}{4}$$
$$=\frac{\left(a^2+2(s^2-2sa+a^2+s^2-s(2s-a)+bc)\right)^2}{4}=\frac{(3a^2-a(a+b+c)+2bc)^2}{4}$$
$$\Rightarrow n_bn_c\geq\frac{(2a^2+2bc-ab-ac)^2}{16R^2}$$
$$\Rightarrow\boxed{\sqrt{n_bn_c}\overset{(\bullet\bullet)}{\geq}\frac{2a^2+2bc-ab-ac}{4R}}\text{ and analogs}$$

$$\left(\because 2a^2+2bc-ab-ac=(y+z)^2+2(x^2+yz)>0\right)\ and\ \sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{2(s-a)}{a}$$

$$=\frac{2s}{4Rrs}.\sum_{cyc}ab-6=\frac{s^2+4Rr+r^2}{2Rr}-6\Rightarrow\sum_{cyc}\frac{h_a}{r_a}=\frac{s^2-8Rr+r^2}{2Rr}$$

$$\therefore\frac{\sqrt{n_bn_c}}{h_a}+\frac{\sqrt{n_cn_a}}{h_b}+\frac{\sqrt{n_an_b}}{h_c}\ge\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}$$

$$\Leftrightarrow\boxed{\sum_{cyc}\frac{n_bn_c}{h_a^2}+2\sum_{cyc}\left(\frac{\sqrt{n_bn_c}}{h_bh_c}.n_a\right)\overset{(*)}{\ge}\frac{\left(s^2-8Rr+r^2\right)^2}{4R^2r^2}}$$

$$\text{Now, via }(\bullet),\boxed{\sum_{cyc}\frac{n_bn_c}{h_a^2}\ge}\sum_{cyc}\frac{a^2(c^2-ca+a^2)(a^2-ab+b^2)}{4R^2.4r^2s^2}$$

$$=\frac{\sum_{cyc}a^6-\sum_{cyc}\left(ab(\sum_{cyc}a^4-c^4)\right)+\sum_{cyc}\left(a^2b^2(\sum_{cyc}a^2-c^2)\right)+abc\sum_{cyc}a^3+3a^2b^2c^2}{16R^2r^2s^2}$$

$$\frac{-abc\left((\sum_{cyc}a)(\sum_{cyc}ab)-3abc\right)}{16R^2r^2s^2}$$

$$=\boxed{\frac{2\left(s^6-(16Rr+9r^2)s^4+r^2s^2(96R^2+76Rr+19r^2)-3r^3(4R+r)^3\right)}{16R^2r^2s^2}}\to(\text{i})$$

$$\left(\begin{array}{c}\text{using}\sum_{cyc}a^6=4s^2(s^2-6Rr-3r^2)^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right),\\ \sum_{cyc}a^4=2\sum_{cyc}a^2b^2-16r^2s^2\ and\sum_{cyc}a^2b^2=(s^2+4Rr+r^2)^2-16Rrs^2\end{array}\right)$$

$$\text{Now, via }(\bullet)\text{ and }(\bullet\bullet),\boxed{2\sum_{cyc}\left(\frac{\sqrt{n_bn_c}}{h_bh_c}.n_a\right)}\ge$$

$$2.\sum_{cyc}\left(\frac{2a^2+2bc-ab-ac}{4R}.\frac{b^2-bc+c^2}{2R}.\frac{bc}{\frac{16R^2r^2s^2}{4R^2}}\right)$$

$$=\frac{abc\sum_{cyc}\left((2a-b-c)(b^2-bc+c^2)\right)+2\sum_{cyc}\left(b^2c^2(b^2-bc+c^2)\right)}{16R^2r^2s^2}$$

$$=\frac{2abc\left((\sum_{cyc}a)(\sum_{cyc}ab)-6abc-\left(3abc+(\sum_{cyc}a)(\sum_{cyc}a^2)-(\sum_{cyc}a)(\sum_{cyc}ab)\right)\right)}{16R^2r^2s^2}$$

$$\frac{+(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)-3a^2b^2c^2-\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)}{16R^2r^2s^2}$$

$$=\boxed{\frac{2\left(s^6-(12Rr+r^2)s^4+r^2s^2(48R^2+16Rr-5r^2)-3r^3(4R+r)^3\right)-32Rr^2s^2(R-2r)}{16R^2r^2s^2}}$$

$\to(\text{ii})\ \therefore(\text{i})\ and\ (\text{ii})\Rightarrow\text{LHS of }(*)\ge$

$$\frac{4\left(s^6-(14Rr+5r^2)s^4+r^2s^2(72R^2+46Rr+7r^2)-3r^3(4R+r)^3-32Rr^2s^2(R-2r)\right)}{16R^2r^2s^2}\overset{?}{\geq}$$

$$\frac{(s^2-8Rr+r^2)^2}{4R^2r^2}\Leftrightarrow(2R-7r)s^4+r^2s^2(78R+6r)-3r^2(4R+r)^3\underset{(**)}{\overset{?}{\geq}}0$$

$$\text{Now, }(2R-7r)s^4+r^2s^2(78R+6r)=(2R-4r)s^4-3rs^4+r^2s^2(78R+6r)$$

$$\overset{\text{Gerretsen}}{\geq}(2R-4r)(16Rr-5r^2)s^2-3r(4R^2+4Rr+3r^2)s^2+r^2s^2(78R+6r)$$

$$=r(20R^2-8Rr+17r^2)s^2\overset{\text{Gerretsen}}{\geq}r(20R^2-8Rr+17r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq}3r^2(4R+r)^3\Leftrightarrow 32t^3-93t^2+69t-22\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(17t^2+15t(t-2)+t+11\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{\sqrt{n_bn_c}}{h_a}+\frac{\sqrt{n_cn_a}}{h_b}+\frac{\sqrt{n_an_b}}{h_c}\geq\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\ \forall\,\Delta\,ABC,$$

with equality iff Δ ABC is equilateral (QED)

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru