RMM - Geometry Marathon 1801 - 1900

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Proposed by

Daniel Sitaru – Romania, George Apostolopoulos-Greece

Jafar Nikpour-Iran, Nelson Tunala-Brazil, Murat Oz-Turkiye

Thanasis Gakopoulos-Greece, Zaza Mzhavanadze-Georgia

Elsen Kerimov-Azerbaijan, Ertan Yildirim-Turkiye

Nguyen Van Canh-Vietnam, Mohamed Amine Ben Ajiba-Morocco

Khaled Abd Imouti-Syria, Mehmet Șahin-Turkiye

Nguyen Hung Cuong-Vietnam, Vasile Mircea Popa-Romania

Radu Diaconu-Romania, Tapas Das-India

Ioannis Stampouloglou-Greece, Bogdan Fuștei-Romania

Marian Ursărescu-Romania, Marin Chirciu-Romania

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

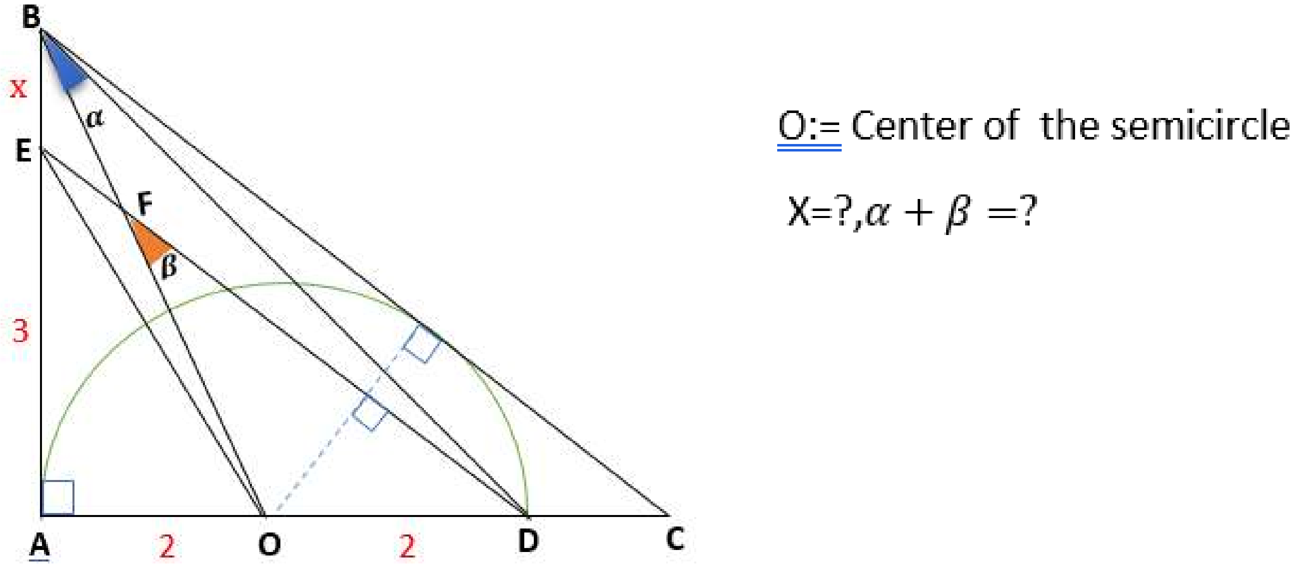
Solutions by

Daniel Sitaru – Romania, Soumava Chakraborty-India

Tapas Das-India, Mirsadix Muzefferov-Azerbaijan

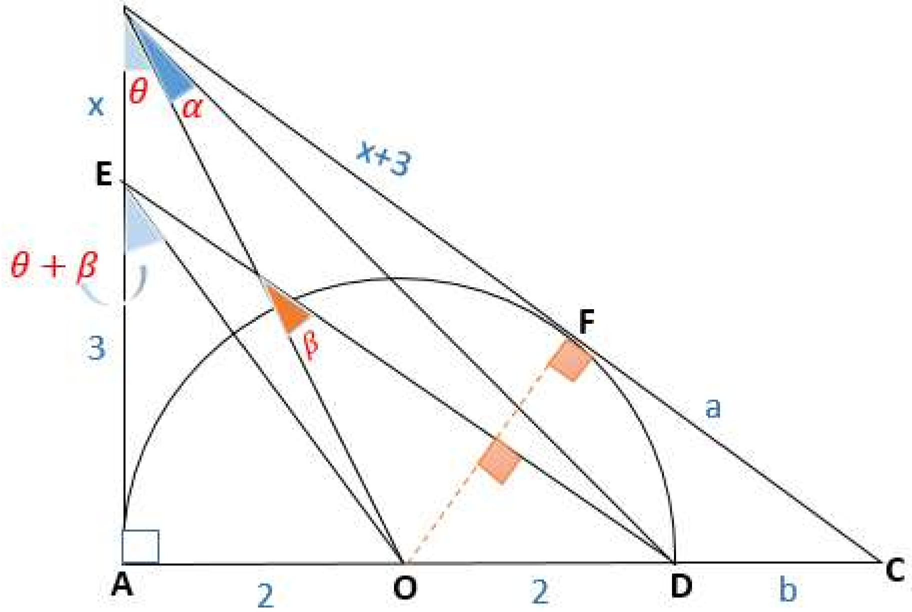
Shirvan Tahirov-Azerbaijan

1801.



Proposed by Jafar Nikpour-Iran

Solution by Mirsadix Muzefferov-Azerbaijan



$$In\ \Delta EAD\ (A = 90°)\ by\ teorem\ Pythagorean$$
$$ED^2 = AD^2 + AE^2 = 4^2 + 3^2 = 25 \Rightarrow ED = 5.$$
$$Also, ED \parallel BC \Rightarrow \Delta AED \sim \Delta ABC$$

$$\frac{5}{(3+x)+a}=\frac{3}{3+x}\Rightarrow x+3=\frac{3a}{2}\quad(1)$$

İn ABC BO *is bisector . Then*

$$\frac{3+x}{2}=\frac{(3+x)+a}{2+b}\Longrightarrow\frac{3a}{4}=\frac{\frac{3a}{2}+a}{2+b}\Longrightarrow b=\frac{4}{3}\quad(2)$$

$$In\ \Delta OFC\ \left(\hat{F}=90^\circ\right)\ by\ Pythagorean$$

$$a^2=(2+b)^2-2^2=\frac{100}{9}-4=\frac{64}{9}\Longrightarrow a=\frac{8}{3}\quad(3)$$

$$Therefore\ (1)\ and\ (3)\Longrightarrow x=1$$

$$In\ \Delta ABO\ \left(\hat{A}=90^\circ\right)\ \ \theta+\alpha=\frac{\pi}{4};$$

$$In\ \Delta AEO\ \left(\hat{A}\right)=90^\circ\Longrightarrow \tan(\theta+\beta)=\frac{4}{3}$$

$$In\ \Delta AEO\ \left(\hat{A}=90^\circ\right)\Longrightarrow \tan\theta=\frac{1}{2};$$

$$\frac{4}{3}=\tan(\theta+\beta)=\frac{\tan\theta+\tan\beta}{1-\tan\theta\tan\beta}=\frac{\frac{1}{2}+\tan\beta}{1-\frac{1}{2}\tan\beta}\Longrightarrow\tan\beta=\frac{1}{2}$$

$$1=\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}=\frac{\frac{1}{2}+\tan\alpha}{1-\frac{1}{2}\tan\alpha}\Longrightarrow\tan\alpha=\frac{1}{3}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1\Longrightarrow\alpha+\beta=\frac{\pi}{4}$$

1802. In $\Delta ABC, G$ –centroid, X,Y,Z –circumradies of $\Delta BCG,\Delta CAG,\Delta ABG$.

Prove that:

$$\frac{\sin A}{X\,m_a}=\frac{\sin B}{Ym_b}=\frac{\sin C}{Zm_c}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$BG=\frac{2}{3}m_b, CG=\frac{2}{3}m_c, AG=\frac{2}{3}m_a\,,$$

$$area\ of\ the\ \Delta\, ABC=F\ then\ [\,BGC]=[CGA]=[AGB]=\frac{F}{3},$$

$$X=\frac{BG.CG.BC}{4[BGC]}=\frac{\frac{2}{3}m_b\frac{2}{3}m_c.a}{\frac{4F}{3}}=\frac{am_bm_c}{3F}, similarly\ Y=\frac{bm_cm_a}{3F}and\ Z=\frac{cm_am_b}{3F}$$

$$\frac{\sin A}{X\,m_a}=\frac{a}{2R}\cdot\frac{3F}{am_am_bm_c}=\frac{3F}{2R}\cdot\frac{1}{m_am_bm_c}$$

$$\frac{\sin B}{Ym_b}=\frac{b}{2R}\cdot\frac{3F}{bm_am_bm_c}=\frac{3F}{2R}\cdot\frac{1}{m_am_bm_c}$$

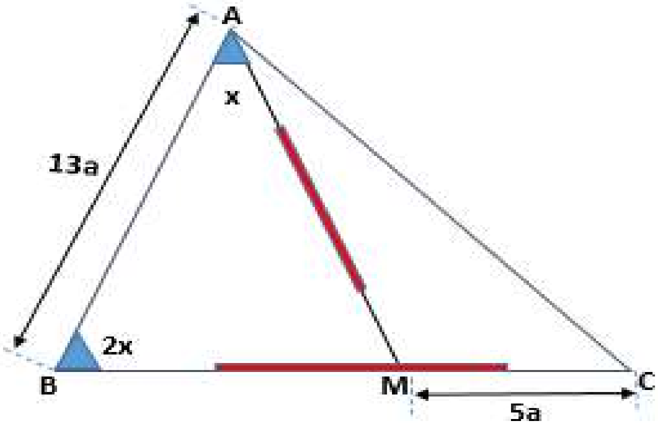
$$\frac{\sin C}{Zm_c}=\frac{c}{2R}\cdot\frac{3F}{cm_am_bm_c}=\frac{3F}{2R}\cdot\frac{1}{m_am_bm_c}$$

$$\frac{\sin A}{X\,m_a}=\frac{\sin B}{Ym_b}=\frac{\sin C}{Zm_c}$$

1803. **In the triangle ABC , the point M (on side BC) is such that:**

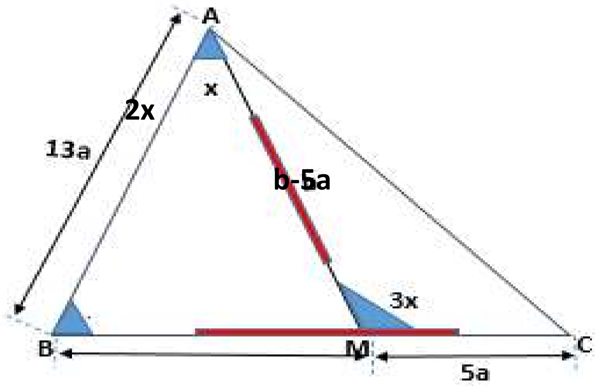
$$MA=BC\,,MC=\left(\frac{5}{13}\right)\cdot AB\,,M\hat{B}A=2\cdot M\hat{A}B=2x$$

Under these conditions , determine the value of x.



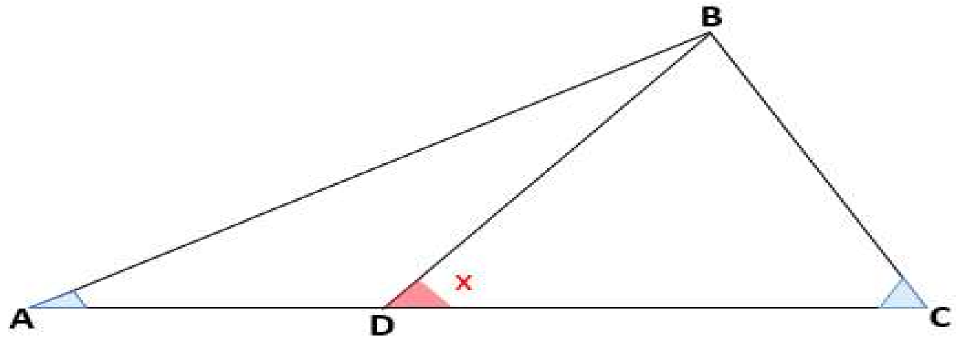
Proposed by Nelson Tunala-Brazil

Solution by Mirsadix Muzefferov-Azerbaijan



Let AM=b.Then BM=b-5a. In ΔABM rule sine

$$\frac{\sin B\hat{M}A}{AB}=\frac{\sin A\hat{B}M}{AM}=\frac{\sin B\hat{A}M}{BM}\Rightarrow\frac{\sin 3x}{13a}=\frac{\sin 2x}{b}=\frac{\sin x}{b-5a}\Rightarrow$$



$$\Rightarrow\frac{a}{b}=\frac{\sin 3x}{13\sin 2x}\quad (1)$$

$$\frac{\sin 2x}{\sin x}=\frac{b}{b-5a}\Rightarrow\frac{a}{b}=\frac{\sin 2x-\sin x}{5\sin 2x}\quad (2)$$

From (1) and (2) we obtain

$$\frac{\sin 2x-\sin x}{5\sin 2x}=\frac{\sin 3x}{13\sin 2x}\Rightarrow 5\sin 3x=13\sin 2x-13\sin x\Rightarrow$$

$$\Rightarrow 5\cdot(3\sin x-4sin^3x)=26\cdot\sin x\cos x-13\sin x\Rightarrow 15-20sin^2x=$$

$$=26\cos x-13\Rightarrow 10cos^2x-13\cos x+4=0$$

Let $\cos x=t$

$$10t^2-13t+4=0\Rightarrow t_1=\frac{1}{2},t_2=\frac{4}{5}$$

$$\cos x=\frac{1}{2}\Rightarrow x=60^\circ$$

it does not satisfy the condition of the problem

$$\cos x=\frac{4}{5}\Rightarrow x=\cos^{-1}\left(\frac{4}{5}\right)\approx 37^\circ$$

Answer: $x=\cos^{-1}\left(\frac{4}{5}\right)\approx 37^\circ$

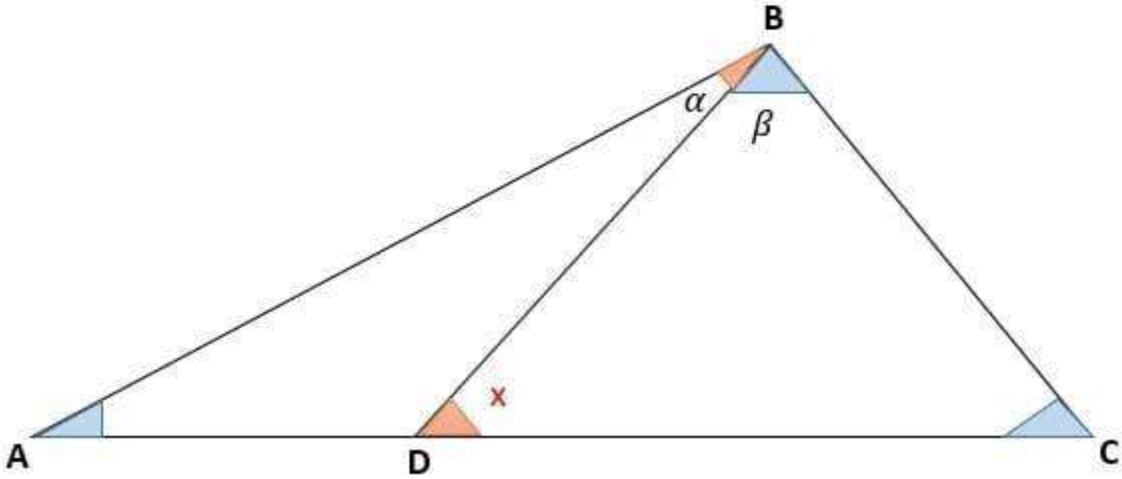
1804.

If $\frac{AD}{DB}=k$ then:

$$\tan x=\frac{(k+1)\tan A\cdot\tan B}{\tan B-k\cdot\tan A}$$

Proposed by Thanasis Gakopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan



$$In\ \Delta ADC\ rule\ sines\ \frac{AD}{\sin\alpha}=\frac{DC}{\sin A}$$

$$In\ \Delta DCB\ also, rule\ sines:\ \frac{DB}{\sin\beta}=\frac{DC}{\sin B}$$

$$\begin{cases}\frac{AD}{\sin\alpha}=\frac{DC}{\sin A}\\ \frac{DB}{\sin\beta}=\frac{DC}{\sin B}\end{cases}\Rightarrow\frac{AD}{\sin\alpha}:\frac{DB}{\sin\beta}=\frac{DC}{\sin A}:\frac{DC}{\sin B}\Rightarrow\frac{AD}{DB}\cdot\frac{\sin\beta}{\sin\alpha}=\frac{\sin B}{\sin A}\Rightarrow k\cdot\frac{\sin\beta}{\sin\alpha}=$$

$$=\frac{\sin B}{\sin A}\Rightarrow\beta=180^\circ-(B+x)\ ;\ \alpha=x-A$$

$$k\cdot\frac{\sin(B+x)}{\sin(x-A)}=\frac{\sin B}{\sin A}\Rightarrow k\cdot\frac{\sin B\cos x+\sin x\cos B}{\sin x\cos A-\sin A\cos x}=\frac{\sin B}{\sin A}\Rightarrow$$

$$\Rightarrow k\cdot\frac{\tan B+\tan x}{\tan B\frac{\cos A}{\cos B}-\frac{\sin A}{\cos B}}=\frac{\sin B}{\sin A}\Rightarrow k\cdot(\tan B+\tan x)=\tan x\cdot\frac{\tan B}{\tan A}-\tan B$$

$$(k+1)\tan B=\tan x\left(\frac{\tan B}{\tan A}-k\right)\Rightarrow$$

$$\tan x=\frac{(k+1)\tan B}{\frac{\tan B}{\tan A}-k}=\frac{(k+1)\tan B\cdot\tan A}{\tan B-k\cdot\tan A}$$

1805. In ΔABC the following relationship holds:

$$\frac{n_a+g_b}{r_c}+\frac{n_b+g_c}{r_a}+\frac{n_c+g_a}{r_b}\geq\frac{12r}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$h_a=\frac{bc}{2R},h_b=\frac{ac}{2R},h_c=\frac{ab}{2R}\ and\ r_ar_br_c=s^2r,abc=4Rrs$$

$$\frac{n_a+g_b}{r_c}+\frac{n_b+g_c}{r_a}+\frac{n_c+g_a}{r_b}\ge\frac{h_a+h_b}{r_c}+\frac{h_b+h_c}{r_a}+\frac{h_c+h_a}{r_b}=$$

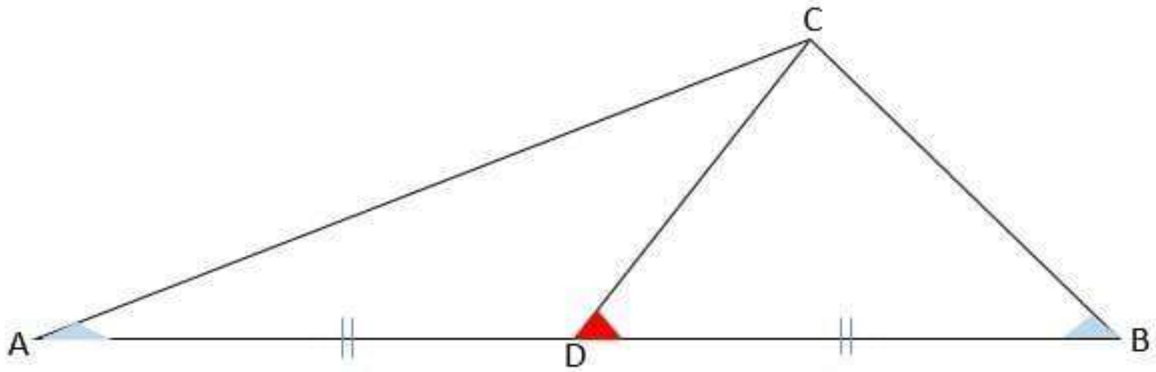
$$=\frac{1}{2R}\left(\frac{bc+ac}{r_c}+\frac{ab+ac}{r_a}+\frac{bc+ba}{r_b}\right)\overset{AM-GM}{\ge}\frac{1}{2R}\left(\frac{2\sqrt{c^2ab}}{r_c}+\frac{2\sqrt{a^2bc}}{r_a}+\frac{2\sqrt{b^2ac}}{r_b}\right)=$$

$$=\frac{1}{R}\left(\frac{\sqrt{c^2ab}}{r_c}+\frac{\sqrt{a^2bc}}{r_a}+\frac{\sqrt{b^2ac}}{r_b}\right)\overset{AM-GM}{\ge}\frac{3}{R}.\left(\frac{\sqrt{c^2ab}}{r_c}.\frac{\sqrt{a^2bc}}{r_a}.\frac{\sqrt{b^2ac}}{r_b}\right)^{\frac{1}{3}}=$$

$$=\frac{3}{R}\left(\frac{a^2b^2c^2}{r_ar_br_c}\right)^{\frac{1}{3}}=\frac{3}{R}\left(\frac{16R^2r^2s^2}{s^2r}\right)^{\frac{1}{3}}\overset{Euler}{\ge}\frac{3}{R}\left(\frac{16(2r)^2r^2}{r}\right)^{\frac{1}{3}}=\frac{3}{R}(64r^3)^{\frac{1}{3}}=\frac{3}{R}(4r)=\frac{12r}{R}$$

$$Equality\ for\ a=b=c$$

1806.

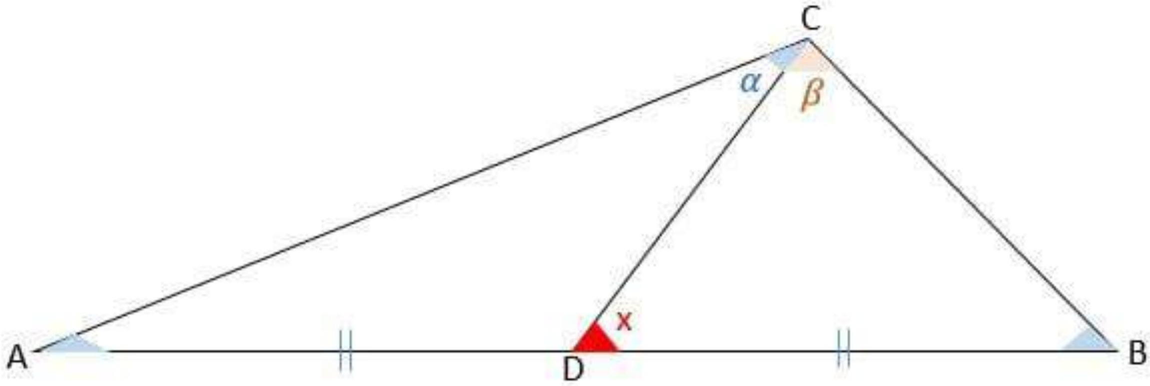


Prove that:

$$\frac{\cos X}{\sin(X+B)}=\frac{\tan B-\tan A}{\tan B+\tan A}\cdot\frac{1}{\sin B}$$

Proposed by Thanasis Gakopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan



$$\left.\begin{array}{r} In\ \Delta ACD\ rule\ sine:\ \dfrac{\sin\alpha}{AD}=\dfrac{\sin A}{DC} \\ In\ \Delta BCD\ rule\ sine:\ \dfrac{\sin\beta}{DB}=\dfrac{\sin B}{DC} \end{array}\right\}\Rightarrow$$

$$\Rightarrow\frac{\sin\alpha}{AD}:\frac{\sin\beta}{DB}=\frac{\sin A}{DC}:\frac{\sin B}{DC}\Rightarrow\frac{\sin\alpha}{\sin\beta}=\frac{\sin A}{\sin B}$$

$$\alpha=x-A\,;\beta=180^\circ-(x+B)$$

$$\frac{\sin(x-A)}{\sin(x+B)}=\frac{\sin A}{\sin B}\Rightarrow\frac{\sin x\cos A-\sin A\cos x}{\sin B\cos x+\sin x\cos B}=\frac{\sin A}{\sin B}$$

$$\sin x\cdot\cos A\cdot\sin B-\cos x\cdot\sin A\cdot\sin B=\sin A\cdot\sin x\cdot\cos B+\sin B\cdot\cos x\cdot\sin A$$

$$2\sin A\cdot\sin B\cdot\cos x=\sin(B-A)\cdot\sin x$$

$$\cos x=\frac{\sin B\cos A-\sin A\cos B}{2\sin A\cdot\sin B}\cdot\sin x=\frac{\tan B-\tan A}{2\tan A\tan B}\cdot\sin x\quad(*)$$

$$\begin{array}{r} In\ \Delta ABC\ rule\ sine\ \dfrac{AB}{\sin(A+B)}=\dfrac{AC}{\sin B} \\ In\ \Delta DBC\ rule\ sine\ \dfrac{DB}{\sin\beta}=\dfrac{DC}{\sin B} \end{array}\Rightarrow\left\{\begin{array}{l} \dfrac{2DB}{\sin(A+B)}=\dfrac{AC}{\sin B} \\ \dfrac{\sin\beta}{DB}=\dfrac{\sin B}{DC} \end{array}\right.\Rightarrow$$

$$\Rightarrow\frac{2\sin\beta}{\sin(A+B)}=\frac{AC}{DC}\quad(1)$$

$$In\ \Delta ACD\Rightarrow\frac{AC}{DC}=\frac{\sin x}{\sin A}\quad(2)$$

From (1) and (2) we have

$$\frac{2\sin\beta}{\sin(A+B)}=\frac{\sin x}{\sin A}\Rightarrow\frac{\sin(A+B)}{2\sin(x+B)}=\frac{\sin A}{\sin x}\quad(**)$$

From (*) and (**) we have

$$\begin{cases} \dfrac{\sin(A+B)}{2\sin(x+B)} = \dfrac{\sin A}{\sin x} \\ \cos x = \dfrac{\tan B - \tan A}{2\tan A\tan B}\cdot\sin x \end{cases} \Rightarrow$$

$$\frac{\sin(A+B)}{2\sin(x+B)}\cdot\cos x = \frac{\sin A}{\sin x}\cdot\frac{\tan B-\tan A}{2\tan A\tan B}\cdot\sin x \Rightarrow$$

$$\Rightarrow \frac{\cos x}{\sin(x+B)} = \frac{\tan B - \tan A}{\frac{\sin A\cos B + \sin B\cos A}{\cos A\cdot\cos B}}\cdot\frac{1}{\sin B} \Rightarrow \frac{\cos x}{\sin(x+B)} =$$

$$= \frac{\tan B-\tan A}{\tan B+\tan A}\cdot\frac{1}{\sin B}\ (proved)$$

1807. In ΔABC the following relationship holds:

$$\frac{\left(\left(\sum_{cyc} a\right)\cdot\left(\sum_{cyc} r_a r_b\right)\cdot\left(\sum_{cyc} r_a\right)\cdot\left(\sum_{cyc}\frac{1}{h_a}\right)\right)}{RF}\cdot\frac{1}{\prod_{cyc} r_a}\cdot r^3 \le 9$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Tapas Das-India

$$\frac{\left(\left(\sum_{cyc} a\right)\cdot\left(\sum_{cyc} r_a r_b\right)\cdot\left(\sum_{cyc} r_a\right)\cdot\left(\sum_{cyc}\frac{1}{h_a}\right)\right)}{RF}\cdot\frac{1}{\prod_{cyc} r_a}\cdot r^3 = \frac{2s.s^2(4R+r).\frac{1}{r}}{RF}\cdot\frac{1}{s^2r}\cdot r^3 =$$

$$= \frac{2(4R+r)}{R} \overset{Euler}{\le} \frac{2\left(4R+\frac{R}{2}\right)}{R} = 9$$

Equality holds for $a=b=c$.

1808.

In any Δ ABC, the following relationship holds :

$$\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{a(b^2+c^2)}+\frac{c^3\cot\frac{A}{2}+a^3\cot\frac{C}{2}}{b(c^2+a^2)}+\frac{a^3\cot\frac{B}{2}+b^3\cot\frac{A}{2}}{c(a^2+b^2)}\ge 3\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A,B,C>0,\ (A+B),(B+C),(C+A) \text{ form sides of a triangle}$$
$$(\because (A+B)+(B+C)>(C+A) \text{ and analogs}) \Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A} \text{ form sides of a triangle with area F (say) and } 16F^2 =$$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \to (1)$$

$$\text{Now, } \forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have: } \frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{a(b^2+c^2)}+\frac{c^3\cot\frac{A}{2}+a^3\cot\frac{C}{2}}{b(c^2+a^2)}+\frac{a^3\cot\frac{B}{2}+b^3\cot\frac{A}{2}}{c(a^2+b^2)}$$

$$=\frac{\frac{ab\cot\frac{C}{2}}{c^2}+\frac{ac\cot\frac{B}{2}}{b^2}}{a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)}+\frac{\frac{bc\cot\frac{A}{2}}{a^2}+\frac{ab\cot\frac{C}{2}}{c^2}}{b^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)}+\frac{\frac{ac\cot\frac{B}{2}}{b^2}+\frac{bc\cot\frac{A}{2}}{a^2}}{c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

$$=\frac{\frac{1}{a^2}}{\frac{1}{b^2}+\frac{1}{c^2}}.\left(\frac{ca\cot\frac{B}{2}}{b^2}+\frac{ab\cot\frac{C}{2}}{c^2}\right)+\frac{\frac{1}{b^2}}{\frac{1}{c^2}+\frac{1}{a^2}}.\left(\frac{ab\cot\frac{C}{2}}{c^2}+\frac{bc\cot\frac{A}{2}}{a^2}\right)$$

$$+\frac{\frac{1}{c^2}}{\frac{1}{a^2}+\frac{1}{b^2}}.\left(\frac{bc\cot\frac{A}{2}}{a^2}+\frac{ca\cot\frac{B}{2}}{b^2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{a^2},y=\frac{1}{b^2},z=\frac{1}{c^2},A=\frac{bc\cot\frac{A}{2}}{a^2},B=\frac{ca\cot\frac{B}{2}}{b^2},C=\frac{ab\cot\frac{C}{2}}{c^2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{via\ (1)\ and\ (2)}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\left(\frac{bc\cot\frac{A}{2}}{a^2}.\frac{ca\cot\frac{B}{2}}{b^2}\right)}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{c^2}{ab}.\cot\frac{A}{2}\cot\frac{B}{2}\right)}\overset{A-G}{\geq}3.\sqrt[6]{\frac{c^2}{ab}.\frac{a^2}{bc}.\frac{b^2}{ca}.\left(\prod_{cyc}\cot\frac{A}{2}\right)^2}=3.\sqrt[6]{\frac{s^6}{(r_ar_br_c)^2}}$$

$$=3.\sqrt[6]{\frac{s^2}{r^2}}\overset{Mitrinovic}{\geq}3.\sqrt[6]{27}=3\sqrt{3}$$

$$\therefore\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{a(b^2+c^2)}+\frac{c^3\cot\frac{A}{2}+a^3\cot\frac{C}{2}}{b(c^2+a^2)}+\frac{a^3\cot\frac{B}{2}+b^3\cot\frac{A}{2}}{c(a^2+b^2)}\geq 3\sqrt{3}$$

$\forall\ \Delta\ ABC,''=''$ iff $\Delta\ ABC$ is equilateral (QED)

1809. In any Δ ABC, the following relationship holds :

$$\frac{\csc^2 A+\csc^2 B+\csc^2 C}{a+b+c}\geq\frac{R}{F}$$

Proposed by Ertan Yildirim-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\csc^2 A+\csc^2 B+\csc^2 C=\sum_{cyc}\frac{4R^2}{a^2}=\frac{4R^2}{16R^2r^2s^2}.\sum_{cyc}a^2b^2$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{4r^2s^2}\overset{?}{\geq}\frac{2sR}{rs}\Leftrightarrow(s^2+4Rr+r^2)^2-16Rrs^2\overset{?}{\geq}8Rrs^2$$

$$\Leftrightarrow s^4-(16Rr-2r^2)s^2+r^2(4R+r)^2\underset{(*)}{\overset{?}{\geq}}0$$

Now, LHS of $(*)\overset{Gerretsen}{\geq}(16Rr-5r^2)s^2-(16Rr-2r^2)s^2+r^2(4R+r)^2$
$=r^2((4R+r)^2-3s^2)\geq 0$ via Trucht (Doucet) $\Rightarrow(*)$ is true

$$\therefore\frac{\csc^2 A+\csc^2 B+\csc^2 C}{a+b+c}\geq\frac{R}{F}\ \forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral (QED)}$$

1810. In ΔABC the following relationship holds:

$$108r^4\leq a^2(s-a)^2+b^2(s-b)^2+c^2(s-c)^2\leq\frac{27R^4}{4}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\sum a^2(s-a)^2 = s^2\sum a^2 - 2s\sum a^3 + \sum a^4 =$$

$$= 2s^2(s^2-r^2-4Rr) - 4s^2(s^2-3r^2-6Rr) +$$
$$+2(s^4-6r^2s^2-8s^2Rr+8Rr^3+16R^2r^2+r^4) =$$

$$= 32R^2r^2+16Rr^3+2r^4-2r^2s^2 \ (1)$$
$$\sum a^2(s-a)^2 = 32R^2r^2+16Rr^3+2r^4-2r^2s^2 \overset{Gerretsen}{\le}$$
$$\le 32R^2r^2-16Rr^3+12r^4$$

$$Now\ \ we\ need\ to\ show:$$
$$32R^2r^2-16Rr^3+12r^4 \le \frac{27R^4}{4}\ or$$

$$27x^4-128x^2+64x-48 \overset{\frac{R}{r}=x\ge 2\ (Euler)}{\ge} 0\ or$$

$$(x-2)[27x^3+x(54x-20)+24] \ge 0\ True$$

$$\sum a^2(s-a)^2 = 32R^2r^2+16Rr^3+2r^4-2r^2s^2 \overset{Gerretsen}{\ge}$$

$$\ge 24R^2r^2+8Rr^3-4r^4 \overset{Euler}{\ge} 96r^4+16r^4-4r^4 = 108r^4$$

Equality holds for $a=b=c$.

1811. In ΔABC the following relationship holds:

$$27r^2 \le h_a^2+h_b^2+h_c^2 \le \frac{27R(R-r)}{2}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$h_a^2+h_b^2+h_c^2 = 4F^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \overset{Steining}{\le} \frac{4F^2}{4r^2} = \frac{r^2s^2}{r^2} = s^2 \overset{Mitrinovic}{\le}$$

$$\le \frac{27}{4}R^2 = \frac{27R}{2}\left(\frac{R}{2}\right) = \frac{27R}{2}\left(R-\frac{R}{2}\right) \overset{Euler}{\le} \frac{27R(R-r)}{2}$$

$$h_a+h_b+h_c\overset{AM-HM}{\geq}\frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=\frac{9}{\frac{1}{r}}=9r\ (1)$$

$$h_a^2+h_b^2+h_c^2\overset{CBS}{\geq}\geq\frac{1}{3}(h_a+h_b+h_c)^2\overset{(1)}{\geq}\frac{1}{3}(9r)^2=27r^2$$

$$Equality\ holds\ for\ a=b=c.$$

1812. In ΔABC the following relationship holds:

$$(\sin A+\sin B+\sin C)^2-\cos(A-B)-\cos(B-C)-\cos(C-A)\leq\frac{15}{4}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\sum\sin A=\frac{s}{R}\ and\ \sum\cos(A-B)=\frac{s^2+r^2+2Rr}{2R^2}-1$$

$$(\sin A+\sin B+\sin C)^2-\cos(A-B)-\cos(B-C)-\cos(C-A)=$$

$$=\left(\frac{s}{R}\right)^2-\frac{s^2+r^2+2Rr}{2R^2}+1=$$

$$=\frac{s^2-r^2-2Rr}{2R^2}+1\overset{Gerrestn}{\leq}\frac{4R^2+2Rr+2r^2}{2R^2}+1\overset{Euler}{\leq}$$

$$\leq\frac{4R^2+R^2+\frac{R^2}{2}}{2R^2}+1=\frac{11}{4}+1=\frac{15}{4}$$

$$Equality\ holds\ for\ A=B=C$$

1813. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin\frac{B}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{B}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of *a* triangle
$(\because (A+B)+(B+C)>(C+A)$ *and analogs*$)\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of *a* triangle with *area* F (*say*) *and* $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now}, \forall x, y, z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have}: \frac{\sin\frac{C}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin\frac{B}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{B}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}+$$

$$\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}+$$

$$\frac{\sin\frac{B}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{A}{2}\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}$$

$$=\frac{\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}}{\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\frac{\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}}{\frac{\sin\frac{B}{2}+\sin\frac{C}{2}}{\sin\frac{A}{2}}}$$

$$+\frac{\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}}{\frac{\sin\frac{C}{2}+\sin\frac{A}{2}}{\sin\frac{B}{2}}}$$

$$=\frac{\sin\frac{C}{2}}{\sin\frac{A}{2}+\sin\frac{B}{2}}\cdot\left(\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}\right)+$$

$$\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}\cdot\left(\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}+\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}\right)+$$

$$\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}+\sin\frac{A}{2}}\cdot\left(\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c}x=\sin\frac{C}{2},y=\sin\frac{A}{2},z=\sin\frac{B}{2},\\ A=\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}},B=\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}},C=\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}\end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{via\ (1)\ and\ (2)}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}\cdot\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}\right)}\overset{A-G}{\geq}$$

$$3.\sqrt{\sqrt[3]{\frac{\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}{\left(\prod_{cyc}\sin\frac{A}{2}\right).\left(\prod_{cyc}\sin\frac{A}{2}\right)}}}\overset{Cesaro}{\geq}3.\sqrt[6]{\frac{8.4R}{r}}\overset{Euler}{\geq}3.\sqrt[6]{64}=6$$

$$\therefore\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin\frac{B}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{B}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}+$$

$$\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}+$$

$$\frac{\sin\frac{B}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{A}{2}\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}\geq 6$$

$\forall\,\Delta\,ABC$,'' ='' iff $\Delta\,ABC$ is equilateral (QED)

1814. In ΔABC the following relationship holds:

$$h_a+h_b+h_c\leq 4R+r$$

Proposed by George Apostolopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$h_a+h_b+h_c=\frac{2S}{a}+\frac{2S}{b}+\frac{2S}{c}$$

On the other hand according to the known formulas for the triangle:

$$S=\frac{ar_br_c}{r_b+r_c}=\frac{br_ar_c}{r_a+r_c}=\frac{cr_ar_b}{r_a+r_b}$$

$$\frac{2S}{a}+\frac{2S}{b}+\frac{2S}{c}=\frac{2r_br_c}{r_b+r_c}+\frac{2r_ar_c}{r_a+r_c}+\frac{2r_ar_b}{r_a+r_b}\overset{A-G}{\leq}\frac{{r_b}^2+{r_c}^2}{r_b+r_c}+\frac{{r_a}^2+{r_c}^2}{r_a+r_c}+$$

$$+\frac{{r_a}^2+{r_b}^2}{r_a+r_b}=\left(r_b+r_c-\frac{2r_br_c}{r_b+r_c}\right)+\left(r_a+r_c-\frac{2r_ar_c}{r_a+r_c}\right)+\left(r_a+r_b-\frac{2r_ar_b}{r_a+r_b}\right)=$$

$$=2(r_a+r_b+r_c)-\left(\frac{2r_br_c}{r_b+r_c}+\frac{2r_ar_c}{r_a+r_c}+\frac{2r_ar_b}{r_a+r_b}\right)=$$

$$=2(r_a+r_b+r_c)-(h_a+h_b+h_c)\Rightarrow 2(h_a+h_b+h_c)\leq 2(r_a+r_b+r_c)$$

$$h_a+h_b+h_c\leq r_a+r_b+r_c=4R+r\quad (proved)$$

Equality holds for $a=b=c$.

1815. **In ΔABC the following relationship holds:**

$$\sum\frac{m_a}{h_b+h_c}+\sum\frac{h_a}{m_b+m_c}+\frac{h_ah_bh_c}{w_aw_bw_c}\frac{R^{2025}}{r^{2025}}\geq 2^{2025}+2\sum\frac{m_a^2}{m_b^2+m_c^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sum\frac{1}{h_a}=\frac{1}{r},\prod h_a\overset{Gm-Hm}{\geq}\left(\frac{1}{\sum\frac{1}{h_a}}\right)^3=27r^3\ and$$

$$\sum m_a\overset{Leunberger}{\leq}(4R+r)\leq\frac{9R}{2}(Euler),$$

$$,\left[\sum\frac{1}{b^2+c^2}\right]\geq\frac{9}{2\sum a^2}(\,Bergstrom)\geq\frac{1}{2R^2}Leibniz$$

$$2\sum\frac{m_a}{h_b+h_c}\overset{m_a\geq h_a}{\geq}\sum\frac{h_a}{h_b+h_c}\overset{Nesbitt}{\geq}\frac{3}{2}\ (1),$$

$$\sum\frac{h_a}{m_b+m_c}\overset{AM-GM}{\geq}3\left(\frac{\prod h_a}{\prod(m_b+m_c)}\right)^{\frac{1}{3}}\overset{AM-GM}{\geq}3(27r^3)^{\frac{1}{3}}\frac{1}{\frac{2\sum m_a}{3}}\geq\frac{9r}{3R}=\frac{3r}{R}$$

$$2\sum\frac{m_a^2}{m_b^2+m_c^2}\leq 2\sum\frac{m_a^2}{h_b^2+h_c^2}=\frac{2(4R^2)}{4}\sum\frac{2(b^2+c^2)-a^2}{a^2c^2+a^2b^2}$$
$$=2R^2\left[\sum\frac{2}{a^2}-\sum\frac{1}{b^2+c^2}\right]\leq 2R^2\left(\frac{2}{4r^2}-\frac{1}{2R^2}\right)(Steining),$$

$$\frac{h_ah_bh_c}{w_aw_bw_c}=\prod\cos\left(\frac{A-B}{2}\right)=\frac{s^2+r^2+2Rr}{8R^2}\geq\frac{9r}{4R}-\frac{r^2}{2R^2}(Gerretsen)$$

$Let\ \frac{R}{r}=x\geq 2\ and\ m=2024,we\ need\ to\ show\ \frac{3}{2}+\frac{3}{x}+\frac{9}{4}x^m-\frac{1}{2}x^{m-1}$
$\geq 2^{m+1}+x^2-1\ or\ 9x^{m+1}-x2^{m+3}+10x-2x^m-4x^3+12\geq 0,$

$$Let\ f(x)=\ 9x^{m+1}-x2^{m+3}+10x-2x^m-4x^3+12,$$
$$f'(x)=9(m+1)x^m-2^{m+3}+10-2mx^{m-1}-12x^2$$
$$=2m(x^m-x^{m-1})+(x^m-12x^2)+(7mx^m+8x^m-2^{m+3})>0\ as\ x\geq 2,$$
$$so\ \ f(x)\ increasing.f(2)=0$$

$$so\ f(x)\geq 0\ \ or\ 9x^{m+1}-x2^{m+3}+10x-2x^m-4x^3+12\geq 0,(true)$$

1816.

In any Δ ABC with n_a, n_b, n_c → Nagel's cevians, the following relationship holds :

$$\sqrt{n_a n_b}+\sqrt{n_b n_c}+\sqrt{n_c n_a} \geq \frac{a^2+b^2+c^2}{2R}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco
Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right) \Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2 \Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s\left(a^2-(b-c)^2\right)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2 \rightarrow (1)$$

$$\text{Now, } n_a \overset{?}{\geq} \frac{b^2-bc+c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\geq} \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2}-1 \overset{?}{\geq} \left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$
$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2} \Leftrightarrow \frac{n_a^2-h_a^2}{h_a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \overset{\text{via } (1)}{\Leftrightarrow}$$
$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow \left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \overset{?}{\geq} \frac{b^2+c^2}{4R^2}\ (\because (b-c)^2 \geq 0)$$
$$\Leftrightarrow 4R^2s^2 \overset{?}{\geq} a^2b^2+c^2a^2 \rightarrow \text{true (strict inequality)} \because 4R^2s^2 \overset{\text{Goldstone}}{\geq} \sum_{cyc} a^2b^2 >$$
$$a^2b^2+c^2a^2 \therefore n_a \overset{(\bullet)}{\geq} \frac{b^2-bc+c^2}{2R} \text{ and analogs}$$
$$\therefore 4R^2.n_bn_c \geq (c^2-ca+a^2)(a^2-ab+b^2)$$
$$=\left((x+y)^2-(x+y)(y+z)+(y+z)^2\right)\left((y+z)^2-(y+z)(z+x)+(z+x)^2\right)$$
$$(x=s-a, y=s-b, z=s-c)=\sum_{cyc}x^4+\sum_{cyc}x^2y^2+2yz(y^2+z^2+yz)+4x^2yz$$
$$=(y^4+z^4+2y^2z^2)+(x^4+y^2z^2+2x^2yz)+(x^2y^2+x^2z^2+2x^2yz)$$
$$+2yz(y^2+z^2) \geq (y^2+z^2)^2+(x^2+yz)^2+x^2(y+z)^2+yz(y+z)^2$$
$$\geq \frac{(y+z)^4}{4}+(x^2+yz)^2+(x^2+yz)(y+z)^2$$
$$=\frac{(y+z)^4+4(x^2+yz)^2+4(x^2+yz)(y+z)^2}{4}=\frac{\left((y+z)^2+2(x^2+yz)\right)^2}{4}$$

$$=\frac{\left(a^2+2\left((s-a)^2+(s-b)(s-c)\right)\right)^2}{4}$$

$$=\frac{\left(a^2+2(s^2-2sa+a^2+s^2-s(2s-a)+bc)\right)^2}{4}=\frac{(3a^2-a(a+b+c)+2bc)^2}{4}$$

$$\Rightarrow n_b n_c \geq \frac{(2a^2+2bc-ab-ac)^2}{16R^2}$$

$$\Rightarrow \boxed{\sqrt{n_b n_c} \overset{(\bullet\bullet)}{\geq} \frac{2a^2+2bc-ab-ac}{4R}} \text{ and analogs}$$

$$(\because 2a^2+2bc-ab-ac=(y+z)^2+2(x^2+yz)>0)$$

$$\therefore \sqrt{n_a n_b}+\sqrt{n_b n_c}+\sqrt{n_c n_a} \geq \sum_{cyc}\frac{2a^2+2bc-ab-ac}{4R}=\frac{a^2+b^2+c^2}{2R}\ \forall \Delta\, ABC,$$

with equality iff Δ ABC is equilateral (QED)

1817.

In any Δ ABC with n_a, n_b, n_c
$\to$ Nagel's cevians, the following relationship holds :

$$\frac{\sqrt{m_a n_a}}{h_a}+\frac{\sqrt{m_b n_b}}{h_b}+\frac{\sqrt{m_c n_c}}{h_c} \geq \frac{\sqrt{2}}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-3\sqrt{2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2 \Rightarrow an_a^2=as^2+$$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-s(a^2-(b-c)^2)=as(s-a)+s(b-c)^2$$

$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2 \to (1)$$

$$\text{Now, } n_a \overset{?}{\geq} \frac{b^2-bc+c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\geq} \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2}-1 \overset{?}{\geq} \left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$

$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2} \Leftrightarrow \frac{n_a^2-h_a^2}{h_a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \overset{\text{via }(1)}{\Leftrightarrow}$$

$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$

$$\Leftrightarrow \left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \overset{?}{\geq} \frac{b^2+c^2}{4R^2}\ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 4R^2s^2 \overset{?}{\geq} a^2b^2+c^2a^2 \to \text{true (strict inequality)} \because 4R^2s^2 \overset{\text{Goldstone}}{\geq}$$

$$\sum_{cyc} a^2b^2 > a^2b^2 + c^2a^2 \therefore n_a \geq \frac{b^2 - bc + c^2}{2R} \Rightarrow \frac{m_a n_a}{h_a^2} \overset{Tereshin}{\geq} \frac{\frac{b^2+c^2}{4R} \cdot \frac{b^2-bc+c^2}{2R}}{\frac{b^2c^2}{4R^2}}$$

$$= \frac{(b^2+c^2)(b^2-bc+c^2)}{2b^2c^2} \Rightarrow \frac{m_a n_a}{h_a^2} - 1 - \frac{(b-c)^4}{2b^2c^2} \geq$$

$$\frac{(b^2+c^2)(b^2-bc+c^2) - 2b^2c^2 - (b-c)^4}{2b^2c^2} = \frac{3bc(b-c)^2}{2b^2c^2} = \frac{3(b-c)^2}{2bc}$$

$$> \frac{\sqrt{2}(b-c)^2}{bc} \Rightarrow \frac{m_a n_a}{h_a^2} \geq 1 + \frac{(b-c)^4}{2b^2c^2} + \frac{\sqrt{2}(b-c)^2}{bc} = \left(1 + \frac{(b-c)^2}{\sqrt{2}bc}\right)^2$$

$$\Rightarrow \frac{\sqrt{m_a n_a}}{h_a} \geq 1 + \frac{(b-c)^2}{\sqrt{2}bc} \text{ and analogs} \therefore \frac{\sqrt{m_a n_a}}{h_a} + \frac{\sqrt{m_b n_b}}{h_b} + \frac{\sqrt{m_c n_c}}{h_c} \geq$$

$$3 + \frac{1}{\sqrt{2}} \sum_{cyc} \frac{b^2 + c^2 - 2bc}{bc} = 3 + \frac{1}{\sqrt{2}} \left(\sum_{cyc} \frac{b+c}{a} - 6 \right)$$

$$\therefore \frac{\sqrt{m_a n_a}}{h_a} + \frac{\sqrt{m_b n_b}}{h_b} + \frac{\sqrt{m_c n_c}}{h_c} \geq \frac{\sqrt{2}}{2}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 3 - 3\sqrt{2}$$

$\forall \Delta ABC$, with equality iff ΔABC is equilateral (QED)

1818. In ΔABC the following relationship holds:

$$\sum \frac{m_a}{h_b + h_c} + \sum \frac{h_a}{w_a + w_b} + \frac{m_a m_b m_c}{w_a w_b w_c} \frac{R^{2024}}{r^{2024}} \geq 2^{2024} + 2 \sum \frac{m_a^2}{w_b^2 + w_c^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

Known:

$$\sum \frac{1}{h_a} = \frac{1}{r}, \prod h_a \overset{Gm-Hm}{\geq} \left(\frac{1}{\sum \frac{1}{h_a}} \right)^3 = 27r^3 \text{ and}$$

$$\sum w_a \leq \sum m_a \overset{Leuenberger}{\leq} (4R + r) \leq \frac{9R}{2} (Euler),$$

$$\frac{m_a m_b m_c}{w_a w_b w_c} \geq 1, \left[\sum \frac{1}{b^2 + c^2}\right] \geq \frac{9}{2\sum a^2} (Bergstrom) \geq \frac{1}{2R^2} (Leibniz)$$

$$2\sum \frac{m_a}{h_b + h_c} \overset{m_a \geq h_a}{\geq} \sum \frac{h_a}{h_b + h_c} \overset{Nesbitt}{\geq} \frac{3}{2} \quad (1),$$

$$\sum \frac{h_a}{w_a + w_b} \overset{AM-GM}{\geq} 3\left(\frac{\prod h_a}{\prod(w_a + w_b)}\right)^{\frac{1}{3}} \overset{AM-GM}{\geq} 3(27r^3)^{\frac{1}{3}} \frac{1}{\frac{2\sum w_a}{3}} \geq \frac{9r}{3R} = \frac{3r}{R} \text{ and}$$

$$2\sum\frac{m_a^2}{w_b^2+w_c^2}\le 2\sum\frac{m_a^2}{h_b^2+h_c^2}=\frac{2(4R^2)}{4}\sum\frac{2(b^2+c^2)-a^2}{a^2c^2+a^2b^2}=$$

$$=2R^2\left[\sum\frac{2}{a^2}-\sum\frac{1}{b^2+c^2}\right]\le 2R^2\left(\frac{2}{4r^2}-\frac{1}{4R^2}\right)(Steining)$$

$$Let\ \frac{R}{r}=x\ge 2, we\ \ need\ to\ show$$

$$\frac{3}{2}+\frac{3}{x}+x^{2024}\ge 2^{2024}+x^2-1\ or$$

$$2x^{2025}-2x^3-x\left(2^{2025}-5\right)+6\ge 0$$

$$Let\ f(x)=2x^{2025}-2x^3-x\left(2^{2025}-5\right)+6$$

$$f'(x)=2.(2025x^{2024})-6x^2-\left(2^{2025}-5\right)=$$

$$=(2025x^{2024}-6x^2)+\left(2025x^{2024}-2^{2025}\right)+5\ge 0$$

$$true\ as\ x\ge 2, f(x)\ is\ increasing\ .$$

$$Equality\ for\ x=2\ and$$

$$f(x)\ge f(2) or\ 2x^{2025}-2x^3-x\left(2^{2025}-5\right)+6\ge 0$$

1819. If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x}{y+z}\sin^2 A+\frac{y}{z+x}\sin^2 B+\frac{z}{x+y}\sin^2 C\ge\frac{9}{8}\left(\frac{R}{2r}\right)^{-2}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\frac{x}{y+z}\sin^2 A+\frac{y}{z+x}\sin^2 B+\frac{z}{x+y}\sin^2 C=$$

$$=\frac{1}{4R^2}\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)\overset{Tsintsifas}{\ge}$$

$$\ge\frac{1}{4R^2}\,2\sqrt{3}\,F\overset{Mitrinovic}{\ge}\frac{1}{4R^2}\,2\sqrt{3}r.3\sqrt{3}r=\frac{9}{8}\left(\frac{2r}{R}\right)^2=\frac{9}{8}\left(\frac{R}{2r}\right)^{-2}$$

$$Equality\ for\ A=B=C\ and\ x=y=z.$$

1820. In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{\tan^7\frac{A}{2}}{\sin^4 A}}+\sqrt[3]{\frac{\tan^7\frac{A}{2}}{\sin^4 A}}+\sqrt[3]{\frac{\tan^7\frac{A}{2}}{\sin^4 A}}\ge\frac{\sqrt[3]{16\sqrt{3}}}{3}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Tapas Das-India

$$\sum \tan\frac{A}{2} = \frac{4R+r}{s} \overset{Doucet}{\geq} \sqrt{3}\ (1)\ and\ \sum \sin A = \frac{s}{R} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}}{2}\ (2)$$

$$\sqrt[3]{\frac{\tan^7\frac{A}{2}}{\sin^4 A}} + \sqrt[3]{\frac{\tan^7\frac{A}{2}}{\sin^4 A}} + \sqrt[3]{\frac{\tan^7\frac{A}{2}}{\sin^4 A}} = \sum \sqrt[3]{\frac{\tan^7\frac{A}{2}}{\sin^4 A}} = \sum \frac{\left(\tan\frac{A}{2}\right)^{\frac{7}{3}}}{(\sin A)^{\frac{4}{3}}} \overset{Radon}{\geq}$$

$$\geq \frac{\left(\sum \tan\frac{A}{2}\right)^{\frac{7}{3}}}{(\sum \sin A)^{\frac{4}{3}}} \overset{(1)\&(2)}{\geq} \frac{(\sqrt{3})^{\frac{7}{3}}}{\left(\frac{3\sqrt{3}}{2}\right)^{\frac{4}{3}}} = \left(\frac{27\sqrt{3}.16}{729}\right)^{\frac{1}{3}} = \frac{\sqrt[3]{16\sqrt{3}}}{3}$$

$$Equality\ for\ A = B = C.$$

1821. **In ΔABC the following relationship holds:**

$$\sum \frac{\cos^n(B-C)}{h_a^n} \geq 3\left(\frac{2}{3R}\right)^n, n \in N$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a\cos(B-C) = 2R\sin A\cos(B-C) = 2R\sin\{\pi-(B+C)\}\cos(B-C)$$
$$= 2R\sin(B+C)\cos(B-C) = R(\sin 2B + \sin 2C)\ (1)$$

$$\sum \frac{\cos^n(B-C)}{h_a^n} = \sum \frac{(a\cos(B-C))^n}{(2F)^n} \overset{CBS}{\geq} \frac{1}{(2F)^n}\cdot\frac{1}{3^{n-1}}\left(\sum a\cos(B-C)\right)^n \overset{(1)}{=}$$

$$= \frac{1}{(2F)^n}\cdot\frac{1}{3^{n-1}}\left(R(\sin 2B + \sin 2C)\right)^n = \frac{1}{(2F)^n}\cdot\frac{2^nR^n}{3^{n-1}}\left(\sum \sin 2A\right)^n =$$

$$= \frac{1}{(2F)^n}\cdot\frac{2^nR^n}{3^{n-1}}\left(4\prod \sin A\right)^n = \frac{1}{(2F)^n}\cdot\frac{2^nR^n}{3^{n-1}}.2^{2n}\left(\frac{sr}{2R^2}\right)^n = \frac{2^n}{3^{n-1}}\cdot\frac{1}{R^n} = 3\left(\frac{2}{3R}\right)^n$$

Equality holds for $a = b = c$.

1822. In ΔABC the following relationship holds:

$$\frac{r_a}{h_a-r}+\frac{r_b}{h_b-r}+\frac{r_c}{h_c-r}\ge\frac{9}{2}$$

Proposed by Mehmet Şahin-Tukiye

Solution by Daniel Sitaru-Romania

$$\frac{r_a}{h_a-r}+\frac{r_b}{h_b-r}+\frac{r_c}{h_c-r}=\sum_{cyc}\frac{r_a}{h_a-r}=$$

$$=\sum_{cyc}\frac{\frac{F}{s-a}}{\frac{2F}{a}-\frac{F}{s}}=\sum_{cyc}\frac{\frac{1}{s-a}}{\frac{2}{a}-\frac{1}{s}}=\sum_{cyc}\frac{sa}{(2s-a)(s-a)}=s\sum_{cyc}\frac{a}{(b+c)(s-a)}\ge$$

$$\overset{AM-GM}{\ge}3s\cdot\sqrt[3]{\frac{abc}{(s-a)(s-b)(s-c)(a+b)(b+c)(c+a)}}=$$

$$=3s\cdot\sqrt[3]{\frac{4RFs}{s(s-a)(s-b)(s-c)}}\cdot\frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}}\overset{AM-GM}{\ge}$$

$$=3s\cdot\sqrt[3]{\frac{4RFs}{F^2}}\cdot\frac{1}{\frac{a+b+b+c+c+a}{3}}=3s\cdot\sqrt[3]{\frac{4Rs}{F}\cdot\frac{3}{4s}}=$$

$$=9\cdot\sqrt[3]{\frac{4Rs}{rs}}\cdot\frac{1}{4}=\frac{9}{4}\cdot\sqrt[3]{\frac{4R}{r}}\overset{EULER}{\ge}\frac{9}{4}\cdot\sqrt[3]{\frac{8r}{r}}=\frac{9}{2}$$

Equality holds for $a=b=c$.

1823. In ΔABC the following relationship holds:

$$\frac{a+b}{h_c}+\frac{b+c}{h_a}+\frac{c+a}{h_b}\ge 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a+b}{h_c}+\frac{b+c}{h_a}+\frac{c+a}{h_b}=\sum_{cyc}\frac{a+b}{h_c}=\sum_{cyc}\frac{a+b}{\frac{2F}{c}}=$$

$$=\frac{1}{2F}\sum_{cyc}(ac+bc)=\frac{1}{F}\sum_{cyc}ab=\frac{1}{F}\cdot(s^2+r^2+4Rr)\overset{GERRETSEN}{\geq}$$

$$\geq\frac{1}{rs}\cdot(16Rr-5r^2+r^2+4Rr)=\frac{1}{s}\cdot(20R-4r)\overset{EULER}{\geq}$$

$$\geq\frac{1}{s}\cdot\left(20R-4\cdot\frac{R}{2}\right)=\frac{18R}{s}\overset{MITRINOVIC}{\geq}\frac{18R}{\frac{3\sqrt{3}}{2}\cdot R}=4\sqrt{3}$$

Equality holds for $a=b=c$.

1824. In ΔABC the following relationship holds:

$$\frac{h_a+h_b}{a+b}+\frac{h_b+h_c}{b+c}+\frac{h_c+h_a}{c+a}\leq\frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{h_a+h_b}{a+b}+\frac{h_b+h_c}{b+c}+\frac{h_c+h_a}{c+a}=\sum_{cyc}\frac{h_a+h_b}{a+b}=$$

$$=\sum_{cyc}\frac{\frac{2F}{a}+\frac{2F}{b}}{a+b}=2F\sum_{cyc}\frac{\frac{1}{a}+\frac{1}{b}}{a+b}=2F\sum_{cyc}\frac{1}{ab}=$$

$$=2F\cdot\frac{a+b+c}{abc}=2F\cdot\frac{2s}{4RF}=\frac{s}{R}\overset{MITRINOVIC}{\leq}\frac{3\sqrt{3}R}{2}\cdot\frac{1}{R}=\frac{3\sqrt{3}}{2}$$

Equality holds for $a=b=c$.

1825.

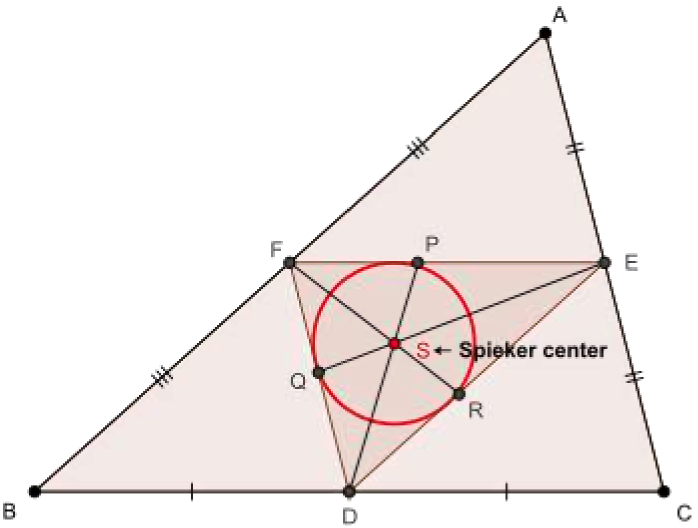
In any Δ ABC with p_a, p_b, p_c

$\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{\sqrt{m_a p_a}}{h_a}+\frac{\sqrt{m_b p_b}}{h_b}+\frac{\sqrt{m_c p_c}}{h_c}\ge\frac{\sqrt{3}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-2\sqrt{3}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS,}\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

Now, $b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2 - bc + 2c^2)^2}{36R^2 \cdot \frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (bc - s(s-a))(b^2+c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq (bc - s(s-a))((2s-a)^2 - 2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$
$$(bc - s(s-a))(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc(s(s-a) + (2s-a)^2) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2} \cdot b^2c^2 - (5s^2 - 5sa + a^2) \cdot bc + s(s-a)(2s-a)^2 \overset{(?)}{\geq} 0}$$

Now, LHS of (?) is a quadratic polynomial with discriminant =

$$(5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2} \cdot s(2s-a)^2$$

$$= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)(12s^2 + 6sa + 5a^2) + 6a^3\right)}{(2s+a)^2} < 0 \ (\because s > a)$$

$\therefore$ (?) is true (strict inequality)

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Rightarrow \frac{m_a p_a}{h_a^2} \overset{\text{Tereshin}}{\geq} \frac{\frac{b^2+c^2}{4R} \cdot \frac{2b^2 - bc + 2c^2}{6R}}{\frac{b^2c^2}{4R^2}}$$

$$\Rightarrow \frac{m_a p_a}{h_a^2} - 1 - \frac{(b-c)^4}{3b^2c^2} \geq \frac{(2b^2 - bc + 2c^2)(b^2+c^2) - 6b^2c^2 - 2(b-c)^4}{6b^2c^2}$$

$$= \frac{7bc(b-c)^2}{6b^2c^2} \geq \frac{2(b-c)^2}{\sqrt{3}bc} \therefore \frac{m_a p_a}{h_a^2} \geq 1 + \frac{(b-c)^4}{3b^2c^2} + \frac{2(b-c)^2}{\sqrt{3}bc} = \left(1 + \frac{(b-c)^2}{\sqrt{3}bc}\right)^2$$

$$\Rightarrow \frac{\sqrt{m_a p_a}}{h_a} \geq 1 + \frac{(b-c)^2}{\sqrt{3}bc} \text{ and analogs} \therefore \frac{\sqrt{m_a p_a}}{h_a} + \frac{\sqrt{m_b p_b}}{h_b} + \frac{\sqrt{m_c p_c}}{h_c} \geq$$

$$3 + \frac{1}{\sqrt{3}} \cdot \sum_{cyc} \frac{b^2 + c^2 - 2bc}{bc} = 3 + \frac{1}{\sqrt{3}} \cdot \left(\sum_{cyc} \frac{b+c}{a} - 6 \right)$$

$$\therefore \frac{\sqrt{m_a p_a}}{h_a} + \frac{\sqrt{m_b p_b}}{h_b} + \frac{\sqrt{m_c p_c}}{h_c} \geq \frac{\sqrt{3}}{3} \left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \right) + 3 - 2\sqrt{3}$$

$\forall \Delta ABC$, with equality iff ΔABC is equilateral (QED)

1826. In ΔABC the following relationship holds:

$$\frac{h_b + h_c}{a} + \frac{h_c + h_a}{b} + \frac{h_a + h_b}{c} \leq 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{h_b + h_c}{a} + \frac{h_c + h_a}{b} + \frac{h_a + h_b}{c} = \sum_{cyc} \frac{h_b + h_c}{a} =$$

$$= \sum_{cyc} \frac{\frac{2F}{b} + \frac{2F}{c}}{a} = 2F \sum_{cyc} \frac{\frac{1}{b} + \frac{1}{c}}{a} = 2F \sum_{cyc} \frac{b+c}{abc} =$$

$$= \frac{2F}{abc} \sum_{cyc} (b+c) = \frac{2F}{4RF} \cdot 2 \sum_{cyc} a = \frac{1}{R} \cdot \sum_{cyc} a =$$

$$= \frac{1}{R} \cdot 2s \overset{MITRINOVIC}{\leq} \frac{1}{R} \cdot 2 \cdot \frac{3\sqrt{3}}{2} \cdot R = 3\sqrt{3}$$

Equality holds for $a = b = c$.

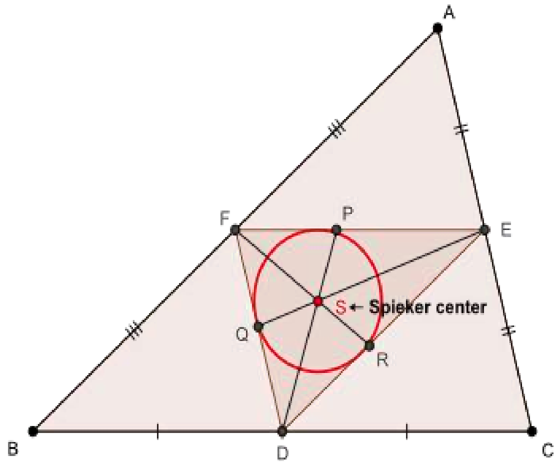
1827.

In any ΔABC with p_a, p_b, p_c $\rightarrow$ Spieker cevians, the following relationship holds:

$$\frac{\sqrt{p_a n_a}}{h_a} + \frac{\sqrt{p_b n_b}}{h_b} + \frac{\sqrt{p_c n_c}}{h_c} \geq \frac{\sqrt{6}}{3} \left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \right) + 3 - 2\sqrt{6}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC } at\ X\ and\ m(\measuredangle BAX) = \alpha\ and\ m(\measuredangle CAX) = \beta\ (say)$$
$$and\ \text{inradius of } \Delta\, DEF = r'(say)$$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$
$$= \frac{\pi}{2} - \frac{A-B}{2}\ and\ m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$\text{Via } (1),(2)\ and \text{ using cosine law on } \Delta\, AFS\ and\ \Delta\, AES, \text{ we arrive } at:$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$
$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$

$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$= 2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$
$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\text{Now, } p_a\ge\frac{2b^2-bc+2c^2}{6R}\Leftrightarrow\frac{p_a^2}{h_a^2}\ge\frac{(2b^2-bc+2c^2)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$
$$\Leftrightarrow\frac{p_a^2-h_a^2}{h_a^2}\ge\left(\frac{2b^2-bc+2c^2}{3bc}-1\right)\left(\frac{2b^2-bc+2c^2}{3bc}+1\right)$$
$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}\frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}}{h_a^2}\ge\frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$
$$\Leftrightarrow\frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2}\ge\frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2}\ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \left(b^2+bc+c^2\right)(s-b)(s-c)$$

$$= \left(b^2+bc+c^2\right)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \left(bc-s(s-a)\right)\left(b^2+c^2\right)+b^2c^2-bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq \left(bc-s(s-a)\right)\left((2s-a)^2-2bc\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$

$$\left(bc-s(s-a)\right)(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc\left(s(s-a)+(2s-a)^2\right) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3-12sa^2-4a^3}{4(s-a)(2s+a)^2}.b^2c^2 - \left(5s^2-5sa+a^2\right).bc + s(s-a)(2s-a)^2 \overset{(▯)}{\geq} 0}$$

Now, LHS of (▯) is a quadratic polynomial with discriminant =

$$\left(5s^2-5sa+a^2\right)^2 - \frac{25s^3-12sa^2-4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2\left(12s^4-18s^3a+5s^2a^2+2sa^3-a^4\right)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)\left(12s^2+6sa+5a^2\right)+6a^3\right)}{(2s+a)^2} < 0\ (\because s > a)$$

$$\therefore (▯) \text{ is true (strict inequality)}$$

$$\therefore p_a \geq \frac{2b^2-bc+2c^2}{6R}\ \text{ and analogs} \to (m)$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s\left(b^2+c^2\right) - bc(2s-a) = an_a^2 + a\left(s^2-s(2s-a)+bc\right) \Rightarrow s\left(b^2+c^2\right) - 2sbc$$

$$= an_a^2 + a\left(as-s^2\right) \Rightarrow s\left(b^2+c^2-a^2-2bc\right) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - s\left(a^2-(b-c)^2\right) = as(s-a) + s(b-c)^2$$

$$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2 \to (l)$$

$$\text{Now, } n_a \overset{?}{\geq} \frac{b^2-bc+c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\geq} \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \overset{?}{\geq} \left(\frac{b^2-bc+c^2}{bc}\right)^2 - 1$$

$$= \frac{(b-c)^2\left(b^2+c^2\right)}{b^2c^2} \Leftrightarrow \frac{n_a^2-h_a^2}{h_a^2} \overset{?}{\geq} \frac{(b-c)^2\left(b^2+c^2\right)}{b^2c^2} \overset{\text{via } (l)}{\Leftrightarrow}$$

$$s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2} \overset{?}{\geq} \frac{(b-c)^2\left(b^2+c^2\right)}{b^2c^2}.\frac{b^2c^2}{4R^2}$$

$$\Leftrightarrow \left(\frac{s}{a} + \frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\geq} \frac{(b-c)^2\left(b^2+c^2\right)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \overset{?}{\geq} \frac{b^2+c^2}{4R^2}\ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 4R^2s^2 \overset{?}{\geq} a^2b^2 + c^2a^2 \rightarrow \text{true (strict inequality)} \because 4R^2s^2 \overset{\text{Goldstone}}{\geq}$$

$$\sum_{\text{cyc}} a^2b^2 > a^2b^2 + c^2a^2 \therefore n_a \geq \frac{b^2 - bc + c^2}{2R} \text{ and analogs} \rightarrow (n)$$

$$\therefore \text{via } (m) \text{ and } (n), \frac{p_a n_a}{h_a^2} \geq \frac{\frac{2b^2 - bc + 2c^2}{6R} \cdot \frac{b^2 - bc + c^2}{2R}}{\frac{b^2c^2}{4R^2}} \Rightarrow \frac{p_a n_a}{h_a^2} - 1 - \frac{2}{3} \cdot \frac{(b-c)^4}{b^2c^2}$$

$$\geq \frac{(2b^2 - bc + 2c^2)(b^2 - bc + c^2) - 3b^2c^2 - 2(b-c)^4}{3b^2c^2} = \frac{5bc(b-c)^2}{3b^2c^2}$$

$$\geq \frac{2\sqrt{6}(b-c)^2}{3bc} \therefore \frac{p_a n_a}{h_a^2} \geq 1 + \frac{2}{3} \cdot \frac{(b-c)^4}{b^2c^2} + \frac{2\sqrt{6}(b-c)^2}{3bc} = \left(1 + \frac{\sqrt{6}(b-c)^2}{3bc}\right)^2$$

$$\Rightarrow \frac{\sqrt{p_a n_a}}{h_a} \geq 1 + \frac{\sqrt{6}(b-c)^2}{3bc} \text{ and analogs} \therefore \frac{\sqrt{p_a n_a}}{h_a} + \frac{\sqrt{p_b n_b}}{h_b} + \frac{\sqrt{p_c n_c}}{h_c} \geq$$

$$3 + \frac{\sqrt{6}}{3} \cdot \sum_{\text{cyc}} \frac{b^2 + c^2 - 2bc}{bc} = 3 + \frac{\sqrt{6}}{3} \cdot \left(\sum_{\text{cyc}} \frac{b+c}{a} - 6\right) \therefore \frac{\sqrt{p_a n_a}}{h_a} + \frac{\sqrt{p_b n_b}}{h_b} + \frac{\sqrt{p_c n_c}}{h_c}$$

$$\geq \frac{\sqrt{6}}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 3 - 2\sqrt{6} \ \forall \Delta ABC,$$

with equality iff ΔABC is equilateral (QED)

1828. In ΔABC the following relationship holds:

$$\frac{a^3}{(b+c)^3 - a^3} + \frac{b^3}{(c+a)^3 - b^3} + \frac{c^3}{(a+b)^3 - c^3} \geq \frac{3}{7}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$LHS = \sum \frac{(b+c)^3}{(b+c)^3 - a^3} - 3 = \sum \frac{1}{1 - \left(\frac{a}{b+c}\right)^3} - 3 \overset{Bergstrom}{\geq}$$

$$\geq \frac{9}{3 - \sum\left(\frac{a}{b+c}\right)^3} - 3 \overset{CBS}{\geq} \frac{9}{3 - \frac{1}{9}\sum\left(\frac{a}{b+c}\right)^3} - 3 \overset{Nesbitt}{\geq}$$

$$\geq \frac{9}{3 - \frac{1}{9} \cdot \frac{27}{8}} - 3 = \frac{24}{7} - 3 = \frac{3}{7}$$

$$Equality\ for\ a = b = c$$

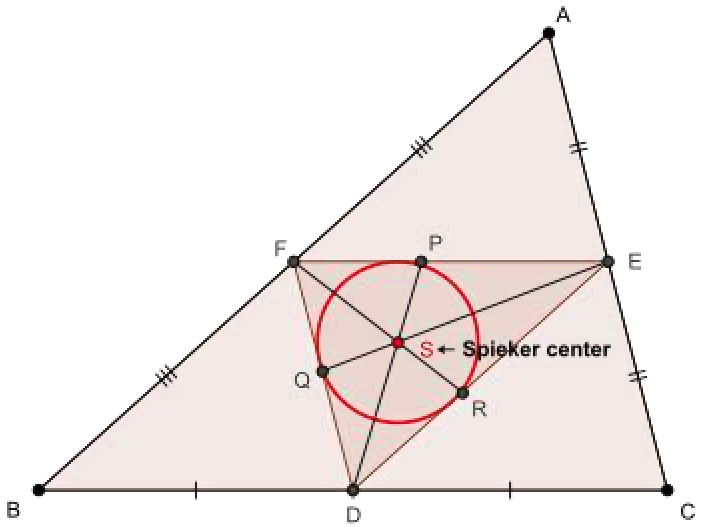
1829. In any Δ ABC with $p_a, p_b, p_c \to$

Spieker cevians, the following relationship holds :

$$\sqrt{p_a p_b} + \sqrt{p_b p_c} + \sqrt{p_c p_a} \geq \frac{2(a^2+b^2+c^2)+ab+bc+ca}{6R}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Proof : Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2} \overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow \mathrm{c}\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\text{ AES}, \mathrm{b}\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now}, [BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a \mathrm{c}\sin\alpha + \frac{1}{2}p_a \mathrm{b}\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now}, b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2 - bc + 2c^2)^2}{36R^2 . \frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (bc - s(s-a))(b^2+c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq (bc - s(s-a))((2s-a)^2 - 2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$

$$(bc - s(s-a))(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc(s(s-a) + (2s-a)^2) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2} . b^2c^2 - (5s^2 - 5sa + a^2) . bc + s(s-a)(2s-a)^2 \overset{(\square)}{\geq} 0}$$

Now, LHS of $(\square)$ is a quadratic polynomial with discriminant =

$$(5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2} . s(2s-a)^2$$

$$= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)(12s^2 + 6sa + 5a^2) + 6a^3\right)}{(2s+a)^2} < 0 \ (\because s > a)$$

$\therefore (\square)$ is true (strict inequality)

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \text{ and analogs}$$

$$\therefore p_b p_c \geq \frac{(2c^2 - ca + 2a^2)(2a^2 - ab + 2b^2)}{36R^2} \overset{?}{\geq} \frac{(4a^2 + 4bc - ab - ac)^2}{144R^2}$$

$$\Leftrightarrow 15a^2(b-c)^2 \overset{?}{\geq} 0 \to \text{true} \therefore p_b p_c \geq \frac{(4a^2+4bc-ab-ac)^2}{144R^2}$$

$$\Rightarrow \boxed{\sqrt{p_b p_c} \geq \frac{4a^2+4bc-ab-ac}{12R}} \text{ and analogs} \therefore \sqrt{p_a p_b}+\sqrt{p_b p_c}+\sqrt{p_c p_a} \geq$$

$$\frac{1}{12R}\cdot\left(4\sum_{cyc} a^2+4\sum_{cyc} bc-\sum_{cyc} ab-\sum_{cyc} ac\right)$$

$$\therefore \sqrt{p_a p_b}+\sqrt{p_b p_c}+\sqrt{p_c p_a} \geq \frac{2(a^2+b^2+c^2)+ab+bc+ca}{6R}$$

$\forall \Delta\, ABC,'' ='' \text{ iff } \Delta\, ABC$ is equilateral (QED)

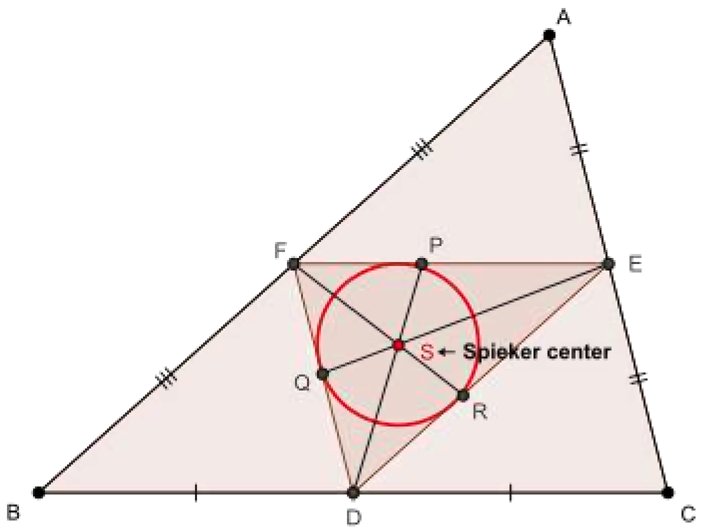
1830.

In any $\Delta\, ABC$ with $p_a, p_b, p_c \to$ Spieker cevians, the following relationship holds :

$$\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c} \geq \frac{2}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)-1$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of $\Delta\, DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4} \Rightarrow r'=\frac{r}{2} \to (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore\boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, }b^3+c^3-abc+a\left(4m_a^2\right)=b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$

$$=(b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$=2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$

$$=(2s+a)\left(b^2-bc+c^2\right)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)\left(b^2-bc+c^2\right)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(b^2-bc+c^2\right)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s + a)\left(s(s - a) + \frac{(b - c)^2}{2} + \frac{a(s - a)}{2}\right) + \frac{s(b - c)^2}{2}$$

$$\therefore \boxed{b^3 + c^3 - abc + a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s + a)\left(\frac{(s - a)(2s + a)}{2} + \frac{(b - c)^2}{2}\right) + \frac{s(b - c)^2}{2}}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s + a)^2}\left(\frac{(s - a)(2s + a)^2}{2} + \frac{(2s + a)(b - c)^2}{2} + \frac{s(b - c)^2}{2}\right)$$

$$= s(s - a) + (b - c)^2\left(\left(\frac{s}{2s + a}\right)^2 + \frac{s}{2s + a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s - a) - \frac{(b - c)^2}{4} + (b - c)^2.\left(\frac{s}{2s + a} + \frac{1}{2}\right)^2$$

$$= s(s - a) + \frac{(b - c)^2}{4}\left(\frac{(4s + a)^2}{(2s + a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s - a) + \frac{s(3s + a)(b - c)^2}{(2s + a)^2}$$

$$\text{Now, } p_a \ge \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \ge \frac{(2b^2 - bc + 2c^2)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \ge \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s - a) + \frac{s(3s + a)(b - c)^2}{(2s + a)^2} - s(s - a) + \frac{s(s - a)(b - c)^2}{a^2}}{h_a^2} \ge \frac{4(b - c)^2(b^2 + bc + c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b - c)^2}{a^2h_a^2(2s + a)^2} \ge \frac{4(b - c)^2(b^2 + bc + c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s - a)(s - b)(s - c)(2s + a)^2} \ge \frac{b^2 + bc + c^2}{9b^2c^2} \ (\because (b - c)^2 \ge 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s - a)(2s + a)^2} \ge (b^2 + bc + c^2)(s - b)(s - c)$$

$$= (b^2 + bc + c^2)(-s(s - a) + bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s - a)(2s + a)^2} \ge (bc - s(s - a))(b^2 + c^2) + b^2c^2 - bcs(s - a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s - a)(2s + a)^2} - b^2c^2 + bcs(s - a) \ge (bc - s(s - a))((2s - a)^2 - 2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s - a)(2s + a)^2} - b^2c^2 + bcs(s - a) \ge$$
$$(bc - s(s - a))(2s - a)^2 - 2b^2c^2 + 2bcs(s - a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s - a)(2s + a)^2} + b^2c^2 + s(s - a)(2s - a)^2 - bc(s(s - a) + (2s - a)^2) \ge 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s - a)(2s + a)^2}.b^2c^2 - (5s^2 - 5sa + a^2).bc + s(s - a)(2s - a)^2 \overset{(?)}{\ge} 0}$$

Now, LHS of (?) is a quadratic polynomial with discriminant =

$$\left(5s^2-5sa+a^2\right)^2-\frac{25s^3-12sa^2-4a^3}{(2s+a)^2}.s(2s-a)^2$$
$$=\frac{-a^2\left(12s^4-18s^3a+5s^2a^2+2sa^3-a^4\right)}{(2s+a)^2}$$
$$=\frac{-a^2(s-a)\left((s-a)\left(12s^2+6sa+5a^2\right)+6a^3\right)}{(2s+a)^2}<0\ (\because s>a)$$
$$\therefore (?)\text{ is true (strict inequality)}$$
$$\therefore p_a\ge\frac{2b^2-bc+2c^2}{6R}\Rightarrow\frac{p_a}{h_a}+\frac{1}{3}\ge\frac{2b^2-bc+2c^2}{3bc}+\frac{1}{3}=\frac{2b^2+2c^2}{3bc}$$
$$\Rightarrow\frac{p_a}{h_a}+\frac{1}{3}\ge\frac{2}{3}\left(\frac{b}{c}+\frac{c}{b}\right)\text{ and analogs }\therefore\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}+1\ge\frac{2}{3}\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)$$
$$=\frac{2}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)\Rightarrow\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\ge\frac{2}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)-1$$
$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1831. In ΔABC non-right angled holds:

$$\frac{\tan A}{\tan B}+\frac{\tan B}{\tan C}+\frac{\tan C}{\tan A}\ge\frac{\sin 2A}{\sin 2B}+\frac{\sin 2B}{\sin 2C}+\frac{\sin 2C}{\sin 2A}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\textit{Lemma } a,b,c>0 \textit{ then }\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}$$

Proof:

$$WLOG\ c=\min(a,b,c),\ now:$$
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\ge\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}.-3=\frac{1}{ab}(a-b)^2+\frac{1}{ac}(a-c)(b-c)\ge$$
$$\ge\left(\frac{1}{ab}-\frac{1}{(a+c)(b+c)}\right)(a-b)^2+\left(\frac{1}{ac}-\frac{1}{(a+c)(a+b)}\right)\times(a-c)(b-c)\ge 0,$$
$$\textit{since } c=\min(a,b,c)$$

$$LHS\overset{lemma}{\ge}\sum\frac{\tan A+\tan B}{\tan B+\tan C}=\sum\frac{\sin(B+A)}{\sin(B+C)}\cdot\frac{\cos C}{\cos A}=$$
$$=\sum\frac{2\sin C\cos C}{2\sin A\cos A}=\sum\frac{\sin 2C}{\sin 2A}\ (Note: A+B+C=\pi)$$

Equality holds for $A=B=C$.

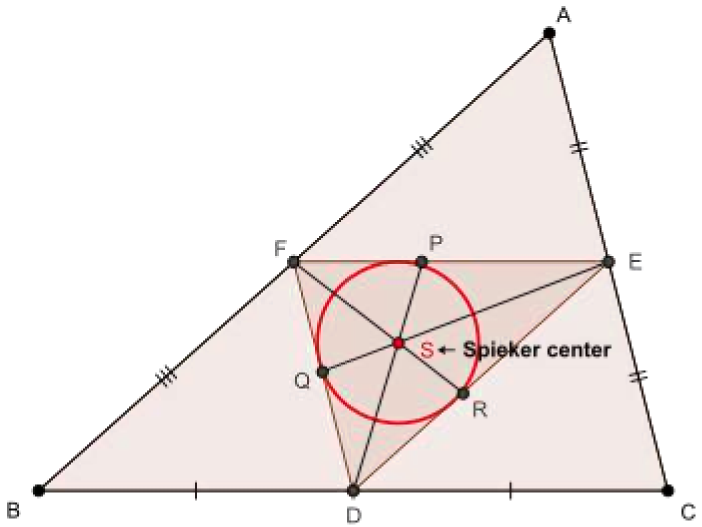
1832.

In any Δ ABC with $p_a\to$ Spieker cevian, the following relationship holds :

$$p_a \geq \frac{2b^2 - bc + 2c^2}{6R}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

Now, $16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$= \frac{\pi}{2} - \frac{A-B}{2}$ and $m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2 - bc + 2c^2)^2}{36R^2 . \frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \big(bc - s(s-a)\big)(b^2+c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq \big(bc - s(s-a)\big)\big((2s-a)^2 - 2bc\big)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$

$$\big(bc - s(s-a)\big)(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc\big(s(s-a) + (2s-a)^2\big) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2} . b^2c^2 - (5s^2 - 5sa + a^2).bc + s(s-a)(2s-a)^2 \overset{(⍰)}{\geq} 0}$$

$$\text{Now, LHS of } (⍰) \text{ is } a \text{ quadratic polynomial with discriminant} =$$

$$(5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2} . s(2s-a)^2$$

$$= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\big((s-a)(12s^2 + 6sa + 5a^2) + 6a^3\big)}{(2s+a)^2} < 0 \ (\because s > a)$$

$$\therefore (⍰) \text{ is true (strict inequality)}$$

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \ \forall \Delta ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$

1833. In ΔABC the following relationship holds:

$$\frac{r_a-r}{h_a-r}+\frac{r_b-r}{h_b-r}+\frac{r_c-r}{h_c-r}\geq 3$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{r_a-r}{h_a-r}+\frac{r_b-r}{h_b-r}+\frac{r_c-r}{h_c-r}=\sum_{cyc}\frac{r_a-r}{h_a-r}=$$

$$=\sum_{cyc}\frac{\frac{F}{s-a}-\frac{F}{s}}{\frac{2F}{a}-\frac{F}{s}}=\sum_{cyc}\frac{\frac{1}{s-a}-\frac{1}{s}}{\frac{2}{a}-\frac{1}{s}}=\sum_{cyc}\frac{as(s-s+a)}{s(2s-a)(s-a)}=$$

$$=\sum_{cyc}\frac{a^2}{(s-a)(b+c)}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{(abc)^2}{(s-a)(s-b)(s-c)(a+b)(b+c)(c+a)}}=$$

$$=3\cdot\sqrt[3]{\frac{(abc)^2\cdot s}{s(s-a)(s-b)(s-c)}}\cdot\frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}}\overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{\frac{(4RF)^2\cdot s}{F^2}}\cdot\frac{3}{a+b+b+c+c+a}=\frac{9}{4s}\cdot\sqrt[3]{16R^2s}\geq$$

$$\overset{MITRINOVIC}{\geq}\frac{9}{4s}\cdot\sqrt[3]{16s\cdot\frac{4}{27}s^2}=\frac{9}{4s}\cdot\frac{4s}{3}=3$$

Equality holds for $a=b=c$.

1834. If in $\Delta ABC, c<b, n\in\mathbb{N}$ then:

$$c^n+h_c^n<b^n+h_b^n$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$we\ need\ to\ show, c^n+h_c^n<b^n+h_b^n$$

$$or\ (c^n-b^n)-(h_b^n-h_c^n)<0$$

$$or\ \ (c^n-b^n)-\frac{1}{2^nR^n}(a^nc^n-a^nb^n)<0$$

$$or\ \ (c^n-b^n)(2^nR^n-a^n)<0$$

$$or\ (2^nR^n-a^n)>0\ (since\ c<b)$$

$$or\ 2^nR^n>a^n=2^n\ R^n\sin^n A$$

$$or\sin^n A<1\ True\ as\ \ \sin A<1$$

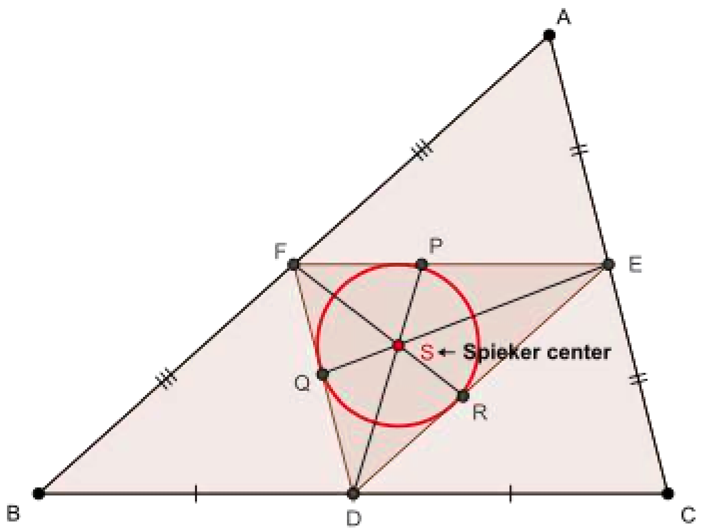
1835.

In any Δ ABC with p_a, p_b, p_c $\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{\sqrt{p_bp_c}}{h_a}+\frac{\sqrt{p_cp_a}}{h_b}+\frac{\sqrt{p_ap_b}}{h_c}\geq\frac{\sqrt{6}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, }16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2 s}\big(ca(s-b)+ab(s-c)\big) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{Now, } b^3+c^3-abc+a\big(4m_a^2\big) = b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$= (b+c)\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2\big)+a\big(b^2+c^2-a^2\big)$$

$$= 2s\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2+bc-a^2\big)$$

$$= (2s+a)\big(b^2-bc+c^2\big)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)\big(b^2-bc+c^2\big)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\big(b^2-bc+c^2\big)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \ge \frac{2b^2-bc+2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \ge \frac{(2b^2-bc+2c^2)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2-h_a^2}{h_a^2} \ge \left(\frac{2b^2-bc+2c^2}{3bc}-1\right)\left(\frac{2b^2-bc+2c^2}{3bc}+1\right)$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \ge \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \ge \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \ge \frac{b^2+bc+c^2}{9b^2c^2}\ (\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \ge (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \ge (bc-s(s-a))(b^2+c^2)+b^2c^2-bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \ge (bc-s(s-a))((2s-a)^2-2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \ge$$
$$(bc-s(s-a))(2s-a)^2-2b^2c^2+2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}+b^2c^2+s(s-a)(2s-a)^2-bc(s(s-a)+(2s-a)^2) \ge 0$$

$$\Leftrightarrow \boxed{\frac{25s^3-12sa^2-4a^3}{4(s-a)(2s+a)^2}.b^2c^2-(5s^2-5sa+a^2).bc+s(s-a)(2s-a)^2 \overset{(\blacksquare)}{\ge} 0}$$

Now, LHS of (■) is a quadratic polynomial with discriminant =

$$\left(5s^2-5sa+a^2\right)^2-\frac{25s^3-12sa^2-4a^3}{(2s+a)^2}.s(2s-a)^2$$
$$=\frac{-a^2\left(12s^4-18s^3a+5s^2a^2+2sa^3-a^4\right)}{(2s+a)^2}$$
$$=\frac{-a^2(s-a)\left((s-a)\left(12s^2+6sa+5a^2\right)+6a^3\right)}{(2s+a)^2}<0\ (\because s>a)$$
$\therefore$ (■) is true (strict inequality)
$$\therefore p_a\geq\frac{2b^2-bc+2c^2}{6R}\text{ and analogs}\to(m)$$
$$\therefore p_bp_c\geq\frac{\left(2c^2-ca+2a^2\right)\left(2a^2-ab+2b^2\right)}{36R^2}\overset{?}{\geq}\frac{\left(4a^2+4bc-ab-ac\right)^2}{144R^2}$$
$$\Leftrightarrow 15a^2(b-c)^2\overset{?}{\geq}0\to\text{true}\therefore p_bp_c\geq\frac{\left(4a^2+4bc-ab-ac\right)^2}{144R^2}$$
$$\Rightarrow\sqrt{p_bp_c}\geq\frac{4a^2+4bc-ab-ac}{12R}\text{ and analogs}\to(n)$$
$$\text{Now, }\frac{\sqrt6}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3=\frac{\sqrt6}{3}\left(\frac{s^2-2Rr+r^2}{2Rr}-6\right)+3$$
$$=\frac{\sqrt6}{3}\left(\frac{s^2-14Rr+r^2}{2Rr}\right)+3\therefore\left(\frac{\sqrt6}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3\right)^2$$
$$=\frac{\left(s^2-14Rr+r^2\right)^2}{6R^2r^2}+9+\frac{\sqrt6}{Rr}\left(s^2-14Rr+r^2\right)$$
$$=\frac{\left(s^2-14Rr+r^2\right)^2+54R^2r^2+6\sqrt6Rr\left(s^2-14Rr+r^2\right)}{6R^2r^2}$$
$$\leq\frac{\left(s^2-14Rr+r^2\right)^2+54R^2r^2+15Rr\left(s^2-14Rr+r^2\right)}{6R^2r^2}$$
$$\left(\because s^2-14Rr+r^2\overset{\text{Gerretsen + Euler}}{\geq}0\text{ and }225>216\Rightarrow15>6\sqrt6\right)$$
$\therefore$ in order to prove the original inequality, it suffices to prove :
$$\boxed{\sum_{cyc}\frac{p_bp_c}{h_a^2}+2\sum_{cyc}\left(\frac{\sqrt{p_bp_c}}{h_bh_c}.p_a\right)\overset{(\blacksquare\blacksquare)}{\geq}\frac{\left(s^2-14Rr+r^2\right)^2+54R^2r^2+15Rr\left(s^2-14Rr+r^2\right)}{6R^2r^2}}$$
$$\text{Now, via (m) and (n), }\sum_{cyc}\frac{p_bp_c}{h_a^2}+2\sum_{cyc}\left(\frac{\sqrt{p_bp_c}}{h_bh_c}.p_a\right)\geq$$
$$\sum_{cyc}\frac{a^2\left(2c^2-ca+2a^2\right)\left(2a^2-ab+2b^2\right)}{36R^2.4r^2s^2}+$$
$$2\sum_{cyc}\left(\frac{4a^2+4bc-ab-ac}{12R}.\frac{2b^2-bc+2c^2}{6R}.\frac{bc}{4r^2s^2}\right)$$
$$=\frac{4\sum_{cyc}a^6-2\sum_{cyc}\left(ab\left(\sum_{cyc}a^4-c^4\right)\right)+12\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)-3abc\sum_{cyc}a^3}{144R^2r^2s^2}$$

$$+\frac{-4\sum_{cyc}a^3b^3+5abc\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)}{144R^2r^2s^2}$$

$$=\frac{8\left(3s^6-\left(36Rr+11r^2\right)s^4+r^2s^2\left(148R^2+116Rr+13r^2\right)-5r^3(4R+r)^3\right)}{144R^2r^2s^2}$$

$$\left(\begin{gathered}\text{using }\sum_{cyc}a^6=4s^2\left(s^2-6Rr-3r^2\right)^2-2\left(\left(s^2+4Rr+r^2\right)^3-24Rrs^2\left(s^2+2Rr+r^2\right)\right),\\ \sum_{cyc}a^4=2\sum_{cyc}a^2b^2-16r^2s^2,\sum_{cyc}a^2b^2=\left(s^2+4Rr+r^2\right)^2-16Rrs^2\text{ and}\\ \sum_{cyc}a^3b^3=\left(s^2+4Rr+r^2\right)^3-24Rrs^2\left(s^2+2Rr+r^2\right)\end{gathered}\right)$$

$$\overset{?}{\geq}\frac{\left(s^2-14Rr+r^2\right)^2+54R^2r^2+15Rr\left(s^2-14Rr+r^2\right)}{6R^2r^2}$$

$$\Leftrightarrow(3R-17r)s^4+rs^2\left(28R^2+155Rr+10r^2\right)-5r^2(4R+r)^3\underset{(\blacksquare\blacksquare\blacksquare)}{\overset{?}{\geq}}0$$

$$\text{Now, }(3R-17r)s^4+rs^2\left(28R^2+155Rr+10r^2\right)$$

$$=(3R-6r)s^4-11rs^4+rs^2\left(28R^2+155Rr+10r^2\right)\overset{\text{Gerretsen}}{\geq}$$

$$(3R-6r)\left(16Rr-5r^2\right)s^2-11r\left(4R^2+4Rr+3r^2\right)s^2+rs^2\left(28R^2+155Rr+10r^2\right)$$

$$=r\left(32R^2+7r^2\right)s^2\overset{?}{\geq}5r^2(4R+r)^3\Leftrightarrow 48t^3-100t^2+13t-10\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(46t^2+2t(t-2)+5\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(\blacksquare\blacksquare\blacksquare)\Rightarrow(\blacksquare\blacksquare)\text{ is true}$$

$$\therefore\frac{\sqrt{p_bp_c}}{h_a}+\frac{\sqrt{p_cp_a}}{h_b}+\frac{\sqrt{p_ap_b}}{h_c}\geq\frac{\sqrt{6}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3$$

$\forall\,\Delta\,ABC$, with equality iff $\Delta\,ABC$ is equilateral (QED)

1836. **In ΔABC the following relationship holds:**

$$e^{3\tan^2\frac{A}{2}}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+e^{3\tan^2\frac{B}{2}}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+e^{3\tan^2\frac{C}{2}}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 2\sqrt{3}\,e$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$$

$$=\left(\sum\tan\frac{A}{2}\right)\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)-\prod\tan\frac{A}{2}=\frac{4R+r}{s}-\frac{r}{s}=\frac{4R}{s}\overset{Mitrinovic}{\geq}$$

$$\geq\frac{4}{s}\cdot\frac{2s}{3\sqrt{3}}=\left(\frac{2}{\sqrt{3}}\right)^3\quad(1)$$

$$\sum 3\tan^2\frac{A}{2}\overset{CBS}{\geq}\left(\sum\tan\frac{A}{2}\right)^2=\left(\frac{4R+r}{s}\right)^2\overset{Doucet}{\geq}3\ (2)$$

$$e^{3\tan^2\frac{A}{2}}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+e^{3\tan^2\frac{B}{2}}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+e^{3\tan^2\frac{C}{2}}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq$$

$$\overset{AM-GM}{\geq}3\left(e^{\sum 3\tan^2\frac{A}{2}}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\right)^{\frac{1}{3}}\overset{(1)\&(2)}{\geq}$$

$$\geq 3\left(e^3\left(\frac{2}{\sqrt{3}}\right)^3\right)^{\frac{1}{3}}=3.e.\frac{2}{\sqrt{3}}=2\sqrt{3}\,e$$

Equality holds for $A=B=C$

1837. In ΔABC the following relationship holds:

$$\pi^{3\tan^2\frac{A}{2}}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\pi^{3\tan^2\frac{B}{2}}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\pi^{3\tan^2\frac{C}{2}}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 12\pi$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\prod\csc\frac{A}{2}=\frac{4R}{r}\overset{Euler}{\geq}8\ (1)\ and$$

$$\sum 3\tan^2\frac{A}{2}\overset{CBS}{\geq}\geq\left(\sum\tan\frac{A}{2}\right)^2=\left(\frac{4R+r}{s}\right)^2\overset{Doucet}{\geq}3\ \ (2)$$

$$\pi^{3\tan^2\frac{A}{2}}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\pi^{3\tan^2\frac{B}{2}}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\pi^{3\tan^2\frac{C}{2}}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)=$$

$$=\sum\pi^{3\tan^2\frac{A}{2}}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)\overset{AM-GM}{\geq}$$

$$\geq 2\sum\pi^{3\tan^2\frac{A}{2}}\sqrt{\csc\frac{A}{2}\csc\frac{B}{2}}\overset{AM-GM}{\geq}6\left(\pi^{\sum 3\tan^2\frac{A}{2}}\prod\csc\frac{A}{2}\right)^{\frac{1}{3}}\overset{(1)\&(2)}{\geq}6(\pi^3.8)^{\frac{1}{3}}=12\pi$$

Equality holds for $A=B=C$.

1838.

If $n_a, n_b, n_c\rightarrow$ Nagel cevians, $g_a, g_b, g_c\rightarrow$ Gergonne's cevians in Δ ABC, then

$$n_a g_a + n_b g_b + n_c g_c \ge s^2.\sqrt{1+\frac{16r^2(R-2r)}{s^2R}}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c) \text{ and}$$
$$b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c) \text{ \& adding the two, we get :}$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)$$
$$=2a\left(n_a^2+g_a^2\right)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$$
$$=2\left(n_a^2+g_a^2\right)+a^2-(b-c)^2\Rightarrow 2\left(b^2+c^2\right)-a^2+(b-c)^2=2\left(n_a^2+g_a^2\right)$$
$$\Rightarrow 4m_a^2+(b-c)^2=2\left(n_a^2+g_a^2\right)\Rightarrow 2(b-c)^2+4s(s-a)=2\left(n_a^2+g_a^2\right)$$
$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$
$$\text{Again, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s\left(b^2+c^2\right)-2sbc$$
$$=an_a^2+a\left(as-s^2\right)\Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2\overset{(**)}{=}s\left(s-a+\frac{(b-c)^2}{a}\right)$$
$$\text{Via } (*) \text{ and } (**), g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$
$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$
$$=(s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)\Rightarrow g_a^2\overset{(***)}{=}(s-a)\left(s-\frac{(b-c)^2}{a}\right)$$
$$\therefore (**),(***)\Rightarrow n_a^2g_a^2=s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$
$$=s(s-a)\left(s(s-a)+s.\frac{(b-c)^2}{a}-\frac{(b-c)^2}{a}(s-a)-\frac{(b-c)^4}{a^2}\right)$$
$$=s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}\right)$$
$$=s^2(s-a)^2+(b-c)^2.\frac{4s(s-a)(s-b)(s-c)}{a^2}=s^2(s-a)^2+(b-c)^2.\frac{4F^2}{a^2}$$

$$\Rightarrow n_a^2 g_a^2 = s^2(s-a)^2 + (b-c)^2 . h_a^2 \text{ and analogs} \rightarrow (1) \therefore n_a^2 g_a^2 . n_b^2 g_b^2 = \left(s^2(s-a)^2 + (b-c)^2 . h_a^2\right).\left(s^2(s-b)^2 + (c-a)^2 . h_b^2\right)$$

$$\geq \left(s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b\right)^2$$

$$\therefore \boxed{n_a g_a . n_b g_b \geq s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b} \text{ and analogs} \rightarrow (2)$$

$$\therefore \sum_{cyc}(n_a g_a . n_b g_b) \geq s^2 \sum_{cyc}(s-a)(s-b) + \sum_{cyc}(|b-c||c-a|.h_a h_b)$$

$$= s^2(4Rr + r^2) + \frac{4Rrs}{4R^2}\sum_{cyc}(c.|b-c||c-a|) \overset{\text{Triangle inequality}}{\geq}$$

$$s^2(4Rr + r^2) + \frac{rs}{R}.\left|\sum_{cyc}(c(b-c)(c-a))\right|$$

$$= s^2(4Rr + r^2) + \frac{rs}{R}.\left|\sum_{cyc} bc^2 - 3abc - \sum_{cyc} c^3 + \sum_{cyc} c^2 a\right|$$

$$= s^2(4Rr + r^2) + \frac{rs}{R}.\left|2s(s^2 + 4Rr + r^2) - 24Rrs - 2s(s^2 - 6Rr - 3r^2)\right|$$

$$= s^2(4Rr + r^2) + \frac{2rs^2}{R}.|-2Rr + 4r^2| = s^2(4Rr + r^2) + \frac{2rs^2}{R}.(2Rr - 4r^2)$$

$$\left(\because 2Rr \overset{\text{Euler}}{\geq} 4r^2\right) \therefore \left(\sum_{cyc} n_a g_a\right)^2 \geq \sum_{cyc} n_a^2 g_a^2 + 2s^2(4Rr + r^2) +$$

$$\frac{4rs^2}{R}.(2Rr - 4r^2) \overset{\text{via (1) and analogs}}{=} s^2 \sum_{cyc}(s-a)^2 + 4r^2 s^2 \sum_{cyc} \frac{(b-c)^2}{a^2}$$

$$+2s^2(4Rr + r^2) + \frac{4rs^2}{R}.(2Rr - 4r^2)$$

$$= s^2\left(3s^2 - 4s^2 + 2(s^2 - 4Rr - r^2)\right) + \frac{4r^2 s^2}{16R^2 r^2 s^2}.\sum_{cyc}\left(b^2c^2\left(\sum_{cyc} a^2 - a^2 - 2bc\right)\right)$$

$$+2s^2(4Rr + r^2) + \frac{4rs^2}{R}.(2Rr - 4r^2)$$

$$= s^2(s^2 - 8Rr - 2r^2) + \frac{1}{4R^2}.\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2 b^2\right) - 48R^2 r^2 s^2 - 2\sum_{cyc} a^3 b^3\right)$$

$$+2s^2(4Rr + r^2) + \frac{4rs^2}{R}.(2Rr - 4r^2) = s^2(s^2 - 8Rr - 2r^2) +$$

$$\frac{4r^2}{4R^2}.\left(-s^4 + (12R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3\right) + 2s^2(4Rr + r^2)$$

$$+\frac{4rs^2}{R}.(2Rr - 4r^2)\left(\begin{array}{c}\text{using } \sum_{cyc} a^2 b^2 = (s^2 + 4Rr + r^2)^2 - 16Rrs^2 \text{ and} \\ \sum_{cyc} a^3 b^3 = (s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2)\end{array}\right)$$

$$= s^4 + \frac{r^2(-s^4 + (12R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3) + 4Rrs^2(2Rr - 4r^2)}{R^2} \overset{?}{\geq}$$

$$s^4\left(1 + \frac{16r^2(R - 2r)}{s^2R}\right) = s^4 + \frac{16r^2s^2(R - 2r)}{R}$$

$$\Leftrightarrow s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3 \underset{(\bullet)}{\overset{?}{\leq}} 0$$

$$\text{Now, Rouche} \Rightarrow s^2 - (m - n) \geq 0 \text{ and } s^2 - (m + n) \leq 0, \text{where } m =$$

$$2R^2 + 10Rr - r^2 \text{ and } n = 2(R - 2r).\sqrt{R^2 - 2Rr}$$

$$\therefore \left(s^2 - (m + n)\right)\left(s^2 - (m - n)\right) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \leq 0 \Rightarrow (\bullet) \text{ is true}$$

$$\Rightarrow \left(\sum_{cyc} n_a g_a\right)^2 \geq s^4\left(1 + \frac{16r^2(R - 2r)}{s^2R}\right)$$

$$\therefore n_a g_a + n_b g_b + n_c g_c \geq s^2.\sqrt{1 + \frac{16r^2(R - 2r)}{s^2R}}$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1839. If in ΔABC $n, m \in N, n \geq m + 1$ then:

$$\frac{n_a^n}{m_a^m} + \frac{n_b^n}{m_b^m} + \frac{n_c^n}{m_c^m} \geq 2^m 3^{n-m+1} \frac{r^n}{R^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$m_a^2 = s(s - a) + \frac{1}{4}(b - c)^2 \text{ and } n_a^2 = s(s - a) + \frac{s(b - c)^2}{a}$$

$$\left[\begin{matrix}\text{Reference: Bogdan Fustei, Mohamed Amine Ben Ajiba} \\ \text{ABOUT NAGEL'S AND GERGONNE'S CEVIANS}\end{matrix}\right]$$

$$n_a^2 - m_a^2 = (b - c)^2\left(\frac{s}{a} - \frac{1}{4}\right) = \frac{(b - c)^2(4s - a)}{4a} \geq 0, \text{so } n_a \geq m_a \text{ or } \frac{n_a}{m_a} \geq 1 \quad (1)$$

$$h_a + h_b + h_c \overset{AM-HM}{\geq} \frac{9}{\sum \frac{1}{h_a}} = \frac{9}{\frac{1}{r}} = 9r \quad (2)$$

Case 1: $n = m + 1$

$$We\ need\ to\ show\ \frac{n_a^{m+1}}{m_a^m}+\frac{n_b^{m+1}}{m_b^m}+\frac{n_c^{m+1}}{m_c^m}\ge 2^m3^2\frac{r^{m+1}}{R^m}$$

$$\frac{n_a^{m+1}}{m_a^m}+\frac{n_b^{m+1}}{m_b^m}+\frac{n_c^{m+1}}{m_c^m}=\sum\left(\left(\frac{n_a}{m_a}\right)^m n_a\right)\overset{(1)}{\ge}\sum n_a\ge\sum h_a\overset{(2)}{\ge}9r=$$
$$=3^2r\frac{r^m}{r^m}\overset{Euler}{\ge}3^2\frac{r^{m+1}}{\left(\frac{R}{2}\right)^m}=2^m3^2\frac{r^{m+1}}{R^m}$$

$Case2: n>m+1$

$$We\ need\ to\ show\ \frac{n_a^n}{m_a^m}+\frac{n_b^n}{m_b^m}+\frac{n_c^n}{m_c^m}\ge 2^m3^{n-m+1}\frac{r^n}{R^m}$$

$$\frac{n_a^n}{m_a^m}+\frac{n_b^n}{m_b^m}+\frac{n_c^n}{m_c^m}=\sum\frac{n_a^n}{m_a^m}=\sum\left(\left(\frac{n_a}{m_a}\right)^m(n_a)^{n-m}\right)\ge$$
$$\ge\sum n_a^{n-m}\overset{CBS}{\ge}\frac{1}{3^{n-m-1}}(n_a+n_b+n_c)^{n-m}\ge$$
$$\ge\frac{1}{3^{n-m-1}}(h_a+h_b+h_c)^{n-m}\overset{(1)}{\ge}\frac{1}{3^{n-m-1}}(9r)^{n-m}=$$
$$=3^{n-m+1}.r^{n-m}.\frac{r^m}{r^m}\overset{Euler}{\ge}3^{n-m+1}\frac{r^n}{R^m}2^m=2^m3^{n-m+1}\frac{r^n}{R^m}$$

$$Equality\ holds\ for\ a=b=c$$

1840. **If in $\Delta ABC, n,m\in N, n\ge m+1$ then:**

$$\frac{\cot^n\frac{A}{2}}{\sin^m A}+,\frac{\cot^n\frac{B}{2}}{\sin^m B}+,\frac{\cot^n\frac{C}{2}}{\sin^m C}\ge 2^m3^{\frac{n-m+2}{2}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ \ a\ge b\ge c\ then\ \sin A\ge\sin B\ge\sin C\ \ and\ \cot\frac{A}{2}\le\cot\frac{B}{2}\le\cot\frac{C}{2}$$

$$For\ n=m+1, we\ need\ to\ show\ \frac{cot^{m+1}\frac{A}{2}}{\sin^m A}+,\frac{cot^{m+1}\frac{B}{2}}{\sin^m B}+,\frac{\cot^{m+1}\frac{C}{2}}{\sin^m C}\ge 2^m3^{\frac{3}{2}}$$

$Proof$:

$$\frac{cot^{m+1}\frac{A}{2}}{\sin^m A}+,\frac{cot^{m+1}\frac{B}{2}}{\sin^m B}+,\frac{\cot^{m+1}\frac{C}{2}}{\sin^m C}\overset{Radon}{\ge}\frac{\left(\sum\cot\frac{A}{2}\right)^{m+1}}{(\sum\sin A)^m}=$$

$$=\frac{\left(\frac{s}{r}\right)^{m+1}}{\left(\frac{s}{R}\right)^{m}}=s\left(\frac{R}{r}\right)^{m}\frac{1}{r}\overset{Mitrinovic\ \&\ Euler}{\geq}2^{m}\,3\sqrt{3}r\ \frac{1}{r}=2^{m}3^{\frac{3}{2}}$$

For $n>m+1$

$$\frac{\cot^n\frac{A}{2}}{\sin^m A}+,\frac{\cot^n\frac{B}{2}}{\sin^m B}+,\frac{\cot^n\frac{C}{2}}{\sin^m C}\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum\cot^n\frac{A}{2}\right)\left(\frac{1}{\sin^m A}\right)=$$

$$=\frac{1}{3}\left(\sum\cot^n\frac{A}{2}\right)\left(\frac{1^{m+1}}{\sin^m A}\right)\overset{CBS\ \&\ Radon}{\geq}\frac{1}{3}\cdot\frac{1}{3^{n-1}}\left(\sum\cot\frac{A}{2}\right)^{n}\frac{(1+1+1)^{m+1}}{(\sum\sin A)^{m}}=$$

$$=\frac{1}{3^n}\left(\frac{s}{r}\right)^{n}\left(\frac{R}{s}\right)^{m}3^{m+1}=\frac{1}{3^n}\frac{s^{n-m}R^m}{r^n}3^{m+1}\overset{Euler\ \&\ Mitrinovic}{\geq}$$

$$\geq\frac{1}{3^n}\frac{\left(3\sqrt{3}r\right)^{n-m}(2r)^m}{r^n}3^{m+1}=2^m\frac{(3)^{\frac{3n-3m}{2}}}{3^n}3^{m+1}=2^m3^{\frac{n-m+2}{2}}$$

Equality holds for $A=B=C$

1841.

If $n_a,n_b,n_c\to$ Nagel cevians, $g_a,g_b,g_c\to$ Gergonne's cevians in ΔABC, then

$$\frac{n_ag_a}{h_a}+\frac{n_bg_b}{h_b}+\frac{n_cg_c}{h_c}\geq\sqrt{16R^2+24Rr-31r^2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$ *and*
$b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)$ & adding the two, we get :
$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)$
$=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$
$=2(n_a^2+g_a^2)+a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)$
$\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$
$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$
Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$
$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$
$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$
$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$

$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 \overset{(**)}{=} s\left(s-a+\frac{(b-c)^2}{a}\right)$$

Via $(*)$ and $(**)$, $g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + (b-c+a)(b-c-a) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a}-1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right) \Rightarrow g_a^2 \overset{(***)}{=} (s-a)\left(s-\frac{(b-c)^2}{a}\right)$$

$$\therefore (**), (***) \Rightarrow n_a^2 g_a^2 = s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$

$$= s(s-a)\left(s(s-a) + s.\frac{(b-c)^2}{a} - \frac{(b-c)^2}{a}(s-a) - \frac{(b-c)^4}{a^2}\right)$$

$$= s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2}\right)$$

$$= s^2(s-a)^2 + (b-c)^2.\frac{4s(s-a)(s-b)(s-c)}{a^2} = s^2(s-a)^2 + (b-c)^2.\frac{4F^2}{a^2}$$

$\Rightarrow n_a^2 g_a^2 = s^2(s-a)^2 + (b-c)^2.h_a^2$ and analogs $\to (1)$ $\therefore n_a^2 g_a^2 . n_b^2 g_b^2 =$

$$\left(s^2(s-a)^2 + (b-c)^2.h_a^2\right).\left(s^2(s-b)^2 + (c-a)^2.h_b^2\right)$$

$$\geq \left(s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b\right)^2$$

$\therefore$ $\boxed{n_a g_a . n_b g_b \geq s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b}$ and analogs $\to (2)$

$$\therefore \sum_{cyc}\left(\frac{n_a g_a}{h_a}.\frac{n_b g_b}{h_b}\right) \geq \sum_{cyc}\frac{s^2 ab(s-a)(s-b)}{4r^2s^2} + \sum_{cyc}(|b-c||c-a|) \overset{\text{Triangle inequality}}{\geq}$$

$$\frac{1}{4r^2}.\sum_{cyc} bc(-s^2+sa+bc) + \left|\sum_{cyc}((b-c)(c-a))\right|$$

$$= \frac{1}{4r^2}.\left(-s^2(s^2+4Rr+r^2) + 12Rrs^2 + (s^2+4Rr+r^2)^2 - 16Rrs^2\right) +$$

$$\left|\sum_{cyc} bc - \sum_{cyc} ab - \sum_{cyc} a^2 + \sum_{cyc} ca\right| = \frac{(4R+r)^2+s^2}{4} + \left|\sum_{cyc} a^2 - \sum_{cyc} ab\right|$$

$$\therefore \left(\sum_{cyc}\frac{n_a g_a}{h_a}\right)^2 \geq \sum_{cyc}\frac{n_a^2 g_a^2}{h_a^2} + \frac{(4R+r)^2+s^2}{2} + 2\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)$$

$$\overset{\text{via }(1)}{=}\sum_{cyc}\frac{s^2a^2(s-a)^2}{4r^2s^2}+\sum_{cyc}(b-c)^2+\frac{(4R+r)^2+s^2}{2}+2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$=\frac{1}{4r^2}\cdot\left(s^2\sum_{cyc}a^2-2s\sum_{cyc}a^3+2\sum_{cyc}a^2b^2-16r^2s^2\right)+\frac{(4R+r)^2+s^2}{2}$$

$$+4\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$=\frac{1}{4r^2}\cdot\left(2s^2(s^2-4Rr-r^2)-4s^2(s^2-6Rr-3r^2)+2(s^2+4Rr+r^2)^2\right.$$
$$\left.-32Rrs^2-16r^2s^2\right)$$

$$+\frac{(4R+r)^2+s^2}{2}+4\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$=\frac{(4R+r)^2-s^2}{2}+\frac{(4R+r)^2+s^2}{2}+4(s^2-12Rr-3r^2)\overset{\text{Gerretsen}}{\geq}$$
$$16R^2+8Rr+r^2+4(16Rr-5r^2-12Rr-3r^2)=16R^2+24Rr-31r^2$$

$$\Rightarrow\left(\sum_{cyc}\frac{n_a g_a}{h_a}\right)^2\geq 16R^2+24Rr-31r^2\therefore\frac{n_a g_a}{h_a}+\frac{n_b g_b}{h_b}+\frac{n_c g_c}{h_c}\geq$$
$$\sqrt{16R^2+24Rr-31r^2}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\text{ is equilateral (QED)}$$

1842. **In ΔABC the following relationship holds:**

$$\frac{\cot^7\frac{A}{2}}{\sin^3 A}+\frac{\cot^7\frac{B}{2}}{\sin^3 B}+\frac{\cot^7\frac{C}{2}}{\sin^3 C}\geq 216$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das

$$WLOG\ \ a\geq b\geq c\ then\ \sin A\geq\sin B\geq\sin C\ \ and\ \cot\frac{A}{2}\leq\cot\frac{B}{2}\leq\cot\frac{C}{2}$$

$$\frac{\cot^7\frac{A}{2}}{\sin^3 A}+,\frac{\cot^7\frac{B}{2}}{\sin^3 B}+,\frac{\cot^7\frac{C}{2}}{\sin^3 C}\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum\cot^7\frac{A}{2}\right)\left(\frac{1}{\sin^3 A}\right)=$$

$$=\frac{1}{3}\left(\sum\cot^7\frac{A}{2}\right)\left(\frac{1^4}{\sin^3 A}\right)\overset{CBS\,\&\,Radon}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{1}{3^6}\left(\sum\cot\frac{A}{2}\right)^7\frac{(1+1+1)^4}{(\sum\sin A)^3}=\frac{1}{27}\left(\frac{s}{r}\right)^7\left(\frac{R}{s}\right)^3=$$

$$=\frac{1}{27}\frac{s^4R^3}{r^7}\overset{Euler\,\&\,Mitrinovic}{\geq}\frac{1}{27}\frac{(27r^2)^2(2r)^3}{r^7}=216$$

$$Equality\ holds\ for\ A = B = C$$

1843. In ΔABC the following relationship holds:

$$\frac{n_a g_a}{m_a} + \frac{n_b g_b}{m_b} + \frac{n_c g_c}{m_c} \geq \frac{18r^2}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$m_a^2 = s(s-a) + \frac{1}{4}(b-c)^2 \text{ and } n_a^2 = s(s-a) + \frac{s(b-c)^2}{a}$$

$$\left[\begin{matrix}Reference: Bogdan\ Fustei\ Mohamed\ Amine\ Ben\ Ajiba \\ ABOUT\ NAGEL'S\ GERGONNE'SCEVIANS\end{matrix}\right]$$

$$n_a^2 - m_a^2 = (b-c)^2\left(\frac{s}{a} - \frac{1}{4}\right) = \frac{(b-c)^2(4s-a)}{4a} \geq 0$$

$$so\ n_a \geq m_a\ or\ \frac{n_a}{m_a} \geq 1 \quad (1)$$

$$\frac{n_a g_a}{m_a} + \frac{n_b g_b}{m_b} + \frac{n_c g_c}{m_c} \overset{(1)}{\geq} g_a + g_b + g_c \geq h_a + h_b + h_c \overset{AM-HM}{\geq}$$

$$\geq \frac{9}{\sum \frac{1}{h_a}} = \frac{9}{\frac{1}{r}} = 9r = \frac{9r^2}{r} \overset{Euler}{\geq} \frac{18r^2}{R}$$

$$Equality\ holds\ for\ a = b = c$$

1844.

If $n_a, n_b, n_c \to$ Nagel cevians, $g_a, g_b, g_c \to$ Gergonne's cevians in Δ ABC, then

$$\frac{n_a g_a}{h_a^2} + \frac{n_b g_b}{h_b^2} + \frac{n_c g_c}{h_c^2} \geq \frac{\sqrt{R^2 + 3Rr - r^2}}{r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \text{ and}$$

$$b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)\ \&\ \text{adding the two, we get :}$$

$$(b^2 + c^2)(2s - b - c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2 + c^2)$$
$$= 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2)$$
$$= 2(n_a^2 + g_a^2) + a^2 - (b-c)^2 \Rightarrow 2(b^2 + c^2) - a^2 + (b-c)^2 = 2(n_a^2 + g_a^2)$$

$$\Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2 + g_a^2)$$

$$\Rightarrow n_a^2 + g_a^2 \stackrel{(*)}{=} (b-c)^2 + 2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 \stackrel{(**)}{=} s\left(s - a + \frac{(b-c)^2}{a}\right)$$

$$\text{Via } (*) \text{ and } (**), g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + (b-c+a)(b-c-a) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a)\left(s - a + \frac{a^2-(b-c)^2}{a}\right) \Rightarrow g_a^2 \stackrel{(***)}{=} (s-a)\left(s - \frac{(b-c)^2}{a}\right)$$

$$\therefore (**), (***) \Rightarrow n_a^2 g_a^2 = s(s-a)\left(s - a + \frac{(b-c)^2}{a}\right)\left(s - \frac{(b-c)^2}{a}\right)$$

$$= s(s-a)\left(s(s-a) + s.\frac{(b-c)^2}{a} - \frac{(b-c)^2}{a}(s-a) - \frac{(b-c)^4}{a^2}\right)$$

$$= s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2}\right)$$

$$= s^2(s-a)^2 + (b-c)^2.\frac{4s(s-a)(s-b)(s-c)}{a^2} = s^2(s-a)^2 + (b-c)^2.\frac{4F^2}{a^2}$$

$$\Rightarrow n_a^2 g_a^2 = s^2(s-a)^2 + (b-c)^2.h_a^2 \text{ and analogs} \to (1) \therefore n_a^2 g_a^2.n_b^2 g_b^2 =$$

$$(s^2(s-a)^2 + (b-c)^2.h_a^2).\left(s^2(s-b)^2 + (c-a)^2.h_b^2\right)$$

$$\geq \left(s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b\right)^2$$

$$\therefore \boxed{n_a g_a.n_b g_b \geq s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b} \text{ and analogs} \to (2)$$

$$\text{Now, } \sum_{cyc}\frac{n_a^2 g_a^2}{h_a^4} \stackrel{\text{via (1)}}{=} \sum_{cyc}\frac{a^4 s^2(s-a)^2}{16r^4 s^4} + \sum_{cyc}\frac{a^2(b-c)^2}{4r^2 s^2}$$

$$=\frac{1}{16r^4s^2}\cdot\left(\left(\sum_{cyc}a^2(s-a)\right)^2-2\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right)\right)+\frac{2\sum_{cyc}a^2b^2-16Rrs^2}{4r^2s^2}$$

$$=\frac{\left(2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\right)^2-2\left(-s^2((s^2+4Rr+r^2)^2-16Rrs^2)\right)}{16r^4s^2}$$

$$+\frac{4Rrs^2(s^2+4Rr+r^2)+(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{16r^4s^2}$$

$$+\frac{2s^2((s^2+4Rr+r^2)^2-16Rrs^2)-16Rrs^2}{4r^2s^2}\ \therefore\ \sum_{cyc}\frac{n_a^2g_a^2}{h_a^4}=$$

$$\frac{3s^4+(8R^2-44Rr+14r^2)s^2-r^2(64R^3-16R^2r-20Rr^2-3r^3)}{8r^2s^2}\rightarrow\text{(i)}$$

Again, $$2\sum_{cyc}\left(\frac{n_ag_a}{h_a^2}\cdot\frac{n_bg_b}{h_b^2}\right)\overset{\text{via }(2)}{\geq}2\sum_{cyc}\frac{s^2(s-a)(s-b)+|b-c||c-a|.h_ah_b}{h_a^2h_b^2}$$

$$=\frac{\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right)}{8r^4s^2}+\frac{1}{2r^2s^2}\cdot\sum_{cyc}(ab.|b-c||c-a|)$$

$$\overset{\text{Triangle inequality}}{\geq}\frac{-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{8r^4s^2}+\frac{1}{2r^2s^2}\cdot\left|\sum_{cyc}(ab(b-c)(c-a))\right|$$

$$=\frac{-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{8r^4s^2}$$

$$+\frac{1}{2r^2s^2}\cdot\left|abc\sum_{cyc}a-\sum_{cyc}a^2b^2-abc\sum_{cyc}a+abc\sum_{cyc}a\right|$$

$$=\frac{-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^2-16Rrs^2-8Rrs^2}{2r^2s^2}\left(\because\sum_{cyc}a^2b^2\geq abc\sum_{cyc}a\right)$$

$$\therefore\ 2\sum_{cyc}\left(\frac{n_ag_a}{h_a^2}\cdot\frac{n_bg_b}{h_b^2}\right)\geq$$

$$\frac{5s^4-(68Rr-10r^2)s^2+r^2(64R^3+112R^2r+44Rr^2+5r^3)}{8r^2s^2}\to(\text{ii})$$

$$\therefore \text{ via (i) and (ii)}, \sum_{cyc}\frac{n_a^2g_a^2}{h_a^4}+2\sum_{cyc}\left(\frac{n_ag_a}{h_a^2}\cdot\frac{n_bg_b}{h_b^2}\right)\geq$$

$$\frac{3s^4+(8R^2-44Rr+14r^2)s^2-r^2(64R^3-16R^2r-20Rr^2-3r^3)}{8r^2s^2}$$

$$+\frac{5s^4-(68Rr-10r^2)s^2+r^2(64R^3+112R^2r+44Rr^2+5r^3)}{8r^2s^2}$$

$$=\frac{s^4+(R^2-14Rr+3r^2)s^2+r^2(4R+r)^2}{r^2s^2}\overset{?}{\geq}\frac{R^2+3Rr-r^2}{r^2}$$

$$\Leftrightarrow s^4-(17Rr-4r^2)s^2+r^2(4R+r)^2\underset{(\bullet)}{\overset{?}{\geq}}0 \text{ and } \because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{ in order to prove }(\bullet),\text{ it suffices to prove : LHS of }(\bullet)\geq(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (5R-2r)s^2\underset{(\bullet\bullet)}{\overset{?}{\geq}}r(80R^2-56Rr+8r^2)$$

$$\text{Now, }(5R-2r)s^2\overset{\text{Rouche}}{\geq}(5R-2r)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\geq}r(80R^2-56Rr+8r^2)$$

$$\Leftrightarrow (R-2r)(10R^2-14Rr+3r^2)\underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}}2(R-2r)(5R-2r)\sqrt{R^2-2Rr}$$

$$\text{and }\because R-2r\overset{\text{Euler}}{\geq}0\ \therefore \text{ in order to prove }(\bullet\bullet\bullet),\text{ it suffices to prove :}$$

$$(10R^2-14Rr+3r^2)^2>\left(2(5R-2r)\sqrt{R^2-2Rr}\right)^2\Leftrightarrow r^2(80R^2-52Rr+9r^2)>0$$

$$\Leftrightarrow r^2(54R^2+26R(R-2r)+9r^2)>0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)$$

$$\text{is true}\therefore\left(\sum_{cyc}\frac{n_ag_a}{h_a^2}\right)^2=\sum_{cyc}\frac{n_a^2g_a^2}{h_a^4}+2\sum_{cyc}\left(\frac{n_ag_a}{h_a^2}\cdot\frac{n_bg_b}{h_b^2}\right)\geq\frac{R^2+3Rr-r^2}{r^2}$$

$$\Rightarrow\frac{n_ag_a}{h_a^2}+\frac{n_bg_b}{h_b^2}+\frac{n_cg_c}{h_c^2}\geq\frac{\sqrt{R^2+3Rr-r^2}}{r}$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)

1845. In ΔABC the following relationship holds:

$$\frac{h_a^n}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{h_b^n}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{h_c^n}{\sin\frac{C}{2}\sin\frac{A}{2}}\geq 4\cdot 3^{n+1}\cdot r^n \qquad n\in N$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\prod \sin\frac{A}{2}=\frac{r}{4R}\overset{Euler}{\leq}\frac{1}{8}\ (1)$$

$$\sqrt[3]{h_ah_bh_c}\overset{Gm-Hm}{\geq}\frac{3}{\sum\frac{1}{h_a}}=3r\ \ (2)$$

$$\frac{h_a^n}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{h_b^n}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{h_c^n}{\sin\frac{C}{2}\sin\frac{A}{2}}\overset{Am-Gm}{\geq}$$

$$\geq\frac{3(h_ah_bh_c)^{\frac{n}{3}}}{\left(\prod\sin\frac{A}{2}\right)^{\frac{2}{3}}}\overset{(1)\&(2)}{\geq}3.3^nr^n.4=4.3^{n+1}\cdot r^n$$

$$Equality\ holds\ for\ a=b=c$$

1846. **In ΔABC the following relationship holds:**

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}\geq\frac{27}{32s^2}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ a+b\geq a+c\geq b+c$$

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}=\sum\frac{a}{(b+c)^3}=$$

$$=\sum\left(\frac{a}{b+c}\cdot\frac{1}{(b+c)^2}\right)\overset{Chebyshev}{\geq}\frac{1}{3}\sum\frac{a}{b+c}\sum\frac{1}{(b+c)^2}=$$

$$=\frac{1}{3}\sum\frac{a}{b+c}\sum\frac{1^3}{(b+c)^2}\overset{Radon\ \&Nesbitt}{\geq}\frac{1}{3}\cdot\frac{3}{2}\cdot\frac{(1+1+1)^3}{(4s)^2}=\frac{27}{32s^2}$$

Equality holds for $a=b=c$.

1847. In ΔABC the following relationship holds:

$$\frac{e^a\left(e^{\tan\frac{B}{2}}+e^{\tan\frac{C}{2}}\right)}{e^b+e^c}+\frac{e^b\left(e^{\tan\frac{C}{2}}+e^{\tan\frac{A}{2}}\right)}{e^c+e^a}+\frac{e^c\left(e^{\tan\frac{A}{2}}+e^{\tan\frac{B}{2}}\right)}{e^a+e^b}\ge 3\,e^{\frac{\sqrt{3}}{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$Walter\ Janous'\ inequality:$$
$$Let\ a,b,c\ and\ x,y,z\ be\ positive\ real\ number\ then:$$
$$\frac{x}{y+z}(b+c)+\frac{y}{z+x}(c+a)+\frac{z}{x+y}(a+b)\ge\sqrt{3(ab+bc+ca)}\ \ (1)$$

$$e^{2\sum\tan\frac{A}{2}}=e^{\frac{2(4R+r)}{s}}\overset{Doucet}{\ge} e^{2\sqrt{3}}\ \ (2)$$

$$\frac{e^a\left(e^{\tan\frac{B}{2}}+e^{\tan\frac{C}{2}}\right)}{e^b+e^c}+\frac{e^b\left(e^{\tan\frac{C}{2}}+e^{\tan\frac{A}{2}}\right)}{e^c+e^a}+\frac{e^c\left(e^{\tan\frac{A}{2}}+e^{\tan\frac{B}{2}}\right)}{e^a+e^b}=$$

$$=\sum\frac{e^a}{e^b+e^c}\left(e^{\tan\frac{B}{2}}+e^{\tan\frac{C}{2}}\right)\overset{(1)}{\ge}\sqrt{3\left(e^{\tan\frac{A}{2}}.e^{\tan\frac{B}{2}}+e^{\tan\frac{B}{2}}.e^{\tan\frac{C}{2}}+e^{\tan\frac{A}{2}}.e^{\tan\frac{C}{2}}\right)}\overset{AM-GM\&(2)}{\ge}$$

$$\ge\sqrt{9\left(e^{2\sqrt{3}}\right)^{\frac{1}{3}}}=3\,e^{\frac{\sqrt{3}}{3}}$$

$$Equality\ for\ a=b=c$$

1848. In ΔABC the following relationship holds:

$$\frac{r_a^3}{(b+c)^2}+\frac{r_b^3}{(c+a)^2}+\frac{r_c^3}{(a+b)^2}\ge\frac{27r}{16}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$\frac{(4R+r)^2}{s^2}\ge 3\ (1)\ and\ (4R+r)\overset{Euler}{\ge} 9r\ \ (2)$$

$$\frac{r_a^3}{(b+c)^2}+\frac{r_b^3}{(c+a)^2}+\frac{r_c^3}{(a+b)^2}\overset{Radon}{\ge}\frac{\left(\sum r_a\right)^3}{(4s)^2}=\frac{(4R+r)^3}{16s^2}=$$

$$=\frac{(4R+r)^2}{s^2}\cdot\frac{4R+r}{16}\overset{(1)\&(2)}{\geq}\frac{27r}{16}$$

Equality holds for $a=b=c$.

1849. In ΔABC the following relationship holds:

$$\frac{a^n}{\cos^2\frac{A}{2}}+\frac{b^n}{\cos^2\frac{B}{2}}+\frac{c^n}{\cos^2\frac{C}{2}}\geq 2^{n+2}\cdot 3^{\frac{n}{2}}\cdot r^n, n\in N$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ a^n\geq b^n\geq c^n\ and$$
$$\cos\frac{A}{2}\leq\cos\frac{B}{2}\leq\cos\frac{C}{2},\sum\cos^2\frac{A}{2}=2+\frac{r}{2R}\overset{Euler}{\leq}\frac{9}{4}$$

$$\frac{a^n}{\cos^2\frac{A}{2}}+\frac{b^n}{\cos^2\frac{B}{2}}+\frac{c^n}{\cos^2\frac{C}{2}}\overset{Chebyshev}{\geq}$$
$$\frac{1}{3}\left(\sum a^n\right)\left(\sum\frac{1}{\cos^2\frac{A}{2}}\right)\geq\frac{1}{3.3^{n-1}}(a+b+c)^n\frac{9}{\sum\cos^2\frac{A}{2}}(CBS)\geq$$

$$\geq\frac{2^ns^n.9}{3.3^{n-1}\frac{9}{4}}\geq 2^{n+2}r^n3^{\frac{3n}{2}}.3^{-n}=2^{n+2}3^{\frac{n}{2}}r^n$$

$$Equality\ for\ a=b=c$$

1850. In ΔABC the following relationship holds:

$$\frac{n_ag_a}{sinA}+\frac{n_bg_b}{sinB}+\frac{n_cg_c}{sinC}\geq 18\sqrt{3}r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{n_ag_a}{sinA}+\frac{n_bg_b}{sinB}+\frac{n_cg_c}{sinC}=\sum_{cyc}\frac{h_ah_a}{sinA}=\sum_{cyc}\frac{\frac{2F}{a}\cdot\frac{2F}{a}}{sinA}=$$

$$= 4F^2 \cdot \sum_{cyc} \frac{1}{a^2 sinA} = 4F^2 \cdot \sum_{cyc} \frac{1}{a^2 \cdot \frac{a}{2R}} = 8RF^2 \sum_{cyc} \frac{1^4}{a^3} \overset{RADON}{\geq}$$

$$\geq 8RF^2 \cdot \frac{(1+1+1)^4}{(a+b+c)^3} = 8Rr^2 s^2 \cdot \frac{81}{8s^3} = \frac{81Rr^2}{s} \overset{MITRINOVIC}{\geq}$$

$$\geq \frac{81Rr^2}{\frac{3\sqrt{3}R}{2}} = \frac{54r^2}{\sqrt{3}} = 18\sqrt{3}r^2$$

Equality holds for $a = b = c$.

1851. In $\Delta ABC, n, m \in N;\ n \geq m+1$ then:

$$\frac{\tan^n \frac{A}{2}}{\sin^m A} + \frac{\tan^n \frac{B}{2}}{\sin^m B} + \frac{\tan^n \frac{C}{2}}{\sin^m C} \geq \frac{2^m}{3^{\frac{n+m-2}{2}}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum \tan \frac{A}{2} \overset{Jensen}{\geq} 3\tan\frac{\pi}{6} = \sqrt{3}\ (1)\ and\ \sum \sin A = \frac{s}{R} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}}{2}\ (2)$$

- $for\ n = m+1\ we\ need\ to\ show$ $\frac{tan^{m+1}\frac{A}{2}}{\sin^m A} + \frac{tan^{m+1}\frac{B}{2}}{\sin^m B} + \frac{tan^{m+1}\frac{C}{2}}{\sin^m C} \geq \frac{2^m}{3^{\frac{2m-1}{2}}}$

$$\frac{tan^{m+1}\frac{A}{2}}{\sin^m A} + \frac{tan^{m+1}\frac{B}{2}}{\sin^m B} + \frac{tan^{m+1}\frac{C}{2}}{\sin^m C} \overset{Radon}{\geq} \frac{\left(\sum \tan\frac{A}{2}\right)^{m+1}}{(\sum \sin A)^m} \overset{(1)\&(2)}{\geq} \frac{\left(\sqrt{3}\right)^{m+1}}{\left(\frac{3\sqrt{3}}{2}\right)^m} = \frac{2^m}{3^{\frac{2m-1}{2}}}$$

- $for\ n > m+1\ we\ need\ to\ show$ $\frac{\tan^n \frac{A}{2}}{\sin^m A} + \frac{\tan^n \frac{B}{2}}{\sin^m B} + \frac{\tan^n \frac{C}{2}}{\sin^m C} \geq \frac{2^m}{3^{\frac{n+m-2}{2}}}$

$$\left(\frac{\tan^n \frac{A}{2}}{\sin^m A} + \frac{\tan^n \frac{B}{2}}{\sin^m B} + \frac{\tan^n \frac{C}{2}}{\sin^m C}\right)\left(\sum \sin A\right)^m (1+1+1)^{n-m-1} \overset{Holder}{\geq} \left(\sum \tan\frac{A}{2}\right)^n$$

$$\left(\frac{\tan^n\frac{A}{2}}{\sin^m A}+\frac{\tan^n\frac{B}{2}}{\sin^m B}+\frac{\tan^n\frac{C}{2}}{\sin^m C}\right)\ge\left(\sum\tan\frac{A}{2}\right)^n\frac{1}{(\sum\sin A)^m3^{n-m-1}}\overset{(1)\&(2)}{\ge}$$

$$\ge\frac{\left(\sqrt{3}\right)^n}{\left(\frac{3\sqrt{3}}{2}\right)^m3^{n-m-1}}=\frac{2^m}{3^{\frac{n+m-2}{2}}}$$

1852. In ΔABC the following relationship holds:

$$\frac{r_a}{b+c-r}+\frac{r_b}{c+a-r}+\frac{r_c}{a+b-r}>\frac{27r}{4s-3r}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ \ then\ r_a\ge r_b\ge r_c\ and\ a+b\ge a+c\ge b+c$$

$$\frac{r_a}{b+c-r}+\frac{r_b}{c+a-r}+\frac{r_c}{a+b-r}\ge$$

$$\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum r_a\right)\left(\sum\frac{1}{b+c-r}\right)\overset{Bergstrom}{>}$$

$$>\frac{1}{3}(4R+r)\cdot\frac{(1+1+1)^2}{4s-3r}\overset{Euler}{\ge}\frac{1}{3}\cdot 9r\cdot\frac{9}{4s-3r}=\frac{27r}{4s-3r}$$

1853. In ΔABC the following relationship holds:

$$\frac{a^n}{\sin^2 A}+\frac{b^n}{\sin^2 B}+\frac{c^n}{\sin^2 C}\ge 2^{n+2}\cdot 3^{\frac{n}{2}}\cdot r^n,\qquad n\in N$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$abc=4Rrs\le 4R\frac{R}{2}\frac{3\sqrt{3}R}{2}(Euler\ \&\ Mitrinovic)=R^3\left(\sqrt{3}\right)^3(1)$$

$$and\ \sum\sin^2 A=\frac{a^2+b^2+c^2}{4R^2}\overset{Leibniz}{\le}\frac{9}{4}(2)$$

$$for\ n = 1\ ,we\ need\ to\ show\ \frac{a}{\sin^2 A} + \frac{b}{\sin^2 B} + \frac{c}{\sin^2 C} \ge 8\sqrt{3}r$$

$$\frac{a}{\sin^2 A} + \frac{b}{\sin^2 B} + \frac{c}{\sin^2 C} = 2R\sum \frac{1}{\sin A} = 4R^2 \sum \frac{1}{a} \overset{Am-Gm}{\ge}$$

$$\ge\ 12R^2\left(\frac{1}{abc}\right)^{\frac{1}{3}} \overset{(1)}{\ge} \frac{12R^2}{R\sqrt{3}} = 8\sqrt{3}r\ (Euler)$$

$$for\ n = 2\ we\ need\ to\ show \frac{a^2}{\sin^2 A} + \frac{b^2}{\sin^2 B} + \frac{c^2}{\sin^2 C} \ge 48r^2$$

$$\frac{a^2}{\sin^2 A} + \frac{b^2}{\sin^2 B} + \frac{c^2}{\sin^2 C} = 12R^2 \ge 48r^2(Euler)$$

$$for\ \ n > 2 \text{ we need to show } \frac{a^n}{\sin^2 A} + \frac{b^n}{\sin^2 B} + \frac{c^n}{\sin^2 C} \ge 2^{n+2}3^{\frac{n}{2}}r^n, n \in N$$

$$\left(\frac{a^n}{\sin^2 A} + \frac{b^n}{\sin^2 B} + \frac{c^n}{\sin^2 C}\right)\left(\sum \sin^2 A\right)(1+1+1)^{n-2} \overset{Holder}{\ge} (a+b+c)^n$$

$$\left(\frac{a^n}{\sin^2 A} + \frac{b^n}{\sin^2 B} + \frac{c^n}{\sin^2 C}\right) \overset{(2)}{\ge} 2^n s^n . \frac{4}{9.3^{n-2}} \overset{Mitrinovic}{\ge} 2^n\, 3^{\frac{3n}{2}} . \frac{4r^n}{9.3^{n-2}} = 2^{n+2}3^{\frac{n}{2}}r^n$$

$$Equality\ for\ a = b = c$$

1854. In ΔABC the following relationship holds:

$$\frac{a^n}{\cos\frac{A}{2}} + \frac{b^n}{\cos\frac{B}{2}} + \frac{c^n}{\cos\frac{C}{2}} \ge 2^{n+1}\, 3^{\frac{n+1}{2}}\, r^n, \qquad \forall\, n \in N$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\bullet\ for\ n = 1\ we\ need\ to\ show \frac{a}{\cos\frac{A}{2}} + \frac{b}{\cos\frac{B}{2}} + \frac{c}{\cos\frac{C}{2}} \ge 12r\ ,$$

$$\frac{a}{\cos\frac{A}{2}} + \frac{b}{\cos\frac{B}{2}} + \frac{c}{\cos\frac{C}{2}} = 4R\sum \sin\frac{A}{2} \overset{am-gm}{\ge} 12R\left(\frac{r}{4R}\right)^{\frac{1}{3}} =$$

$$= 12R\left(\frac{r^3}{4Rr^2}\right)^{\frac{1}{3}} \overset{Euler}{\ge} 12R.\frac{r}{R} = 12r$$

• $for\ n=2, we\ need\ to\ show \frac{a^2}{\cos\frac{A}{2}}+\frac{b^2}{\cos\frac{B}{2}}+\frac{c^2}{\cos\frac{C}{2}}\geq 24\sqrt{3}\,r^2$

$$\frac{a^2}{\cos\frac{A}{2}}+\frac{b^2}{\cos\frac{B}{2}}+\frac{c^2}{\cos\frac{C}{2}}\geq\frac{(a+b+c)^2}{\sum\cos\frac{A}{2}}\overset{Jensen}{\geq}$$

$$\geq\frac{4s^2}{3\cos\frac{\pi}{6}}\overset{Mitrinovic}{\geq}4.27r^2.\frac{2}{3\sqrt{3}}=24\sqrt{3}\,r^2$$

• $for\ n>2$ we need to show $\frac{a^n}{\cos\frac{A}{2}}+\frac{b^n}{\cos\frac{B}{2}}+\frac{c^n}{\cos\frac{C}{2}}\geq 2^{n+1}3^{\frac{n+1}{2}}r^n$

$$\left(\frac{a^n}{\cos\frac{A}{2}}+\frac{b^n}{\cos\frac{B}{2}}+\frac{c^n}{\cos\frac{C}{2}}\right)\left(\sum\cos\frac{A}{2}\right)(1+1+1)^{n-2}\overset{Holder}{\geq}(a+b+c)^n$$

$$\left(\frac{a^n}{\cos\frac{A}{2}}+\frac{b^n}{\cos\frac{B}{2}}+\frac{c^n}{\cos\frac{C}{2}}\right)\geq\frac{(2s)^n}{\left(\sum\cos\frac{A}{2}\right)(1+1+1)^{n-2}}\overset{Jensen}{\geq}$$

$$\geq\frac{2^n s^n}{3\cos\left(\frac{\pi}{6}\right).3^{n-2}}\overset{Mitrinovic}{\geq}\frac{2^n 3^{\frac{3n}{2}}r^n}{\frac{3\sqrt{3}}{2}.3^{n-2}}=2^{n+1}3^{\frac{n+1}{2}}r^n$$

$$Equality\ for\ a=b=c$$

1855. In ΔABC the following relationship holds:

$$F\leq\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{4}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$Heron's\ formula: 2\left(\sum a^2b^2\right)-\sum a^4=16F^2 \qquad (1)$$

$$(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2\geq 0\ or$$

$$b^2c^2+c^2a^2+a^2b^2\leq a^4+b^4+c^4\ (2)$$

$$From\ (1): 16F^2=2\left(\sum a^2b^2\right)-\sum a^4\overset{(2)}{\leq}$$

$$\leq 2\left(\sum a^2b^2\right)-\left(\sum a^2b^2\right)=\left(\sum a^2b^2\right) \text{ or}$$

$$F \leq \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{4}$$

Equality for $a=b=c$

1856. **In any acute triangle ABC, the following relationship holds :**

$$\frac{a}{h_a}\cdot\sqrt{\sin\frac{A}{2}}+\frac{b}{h_b}\cdot\sqrt{\sin\frac{B}{2}}+\frac{c}{h_c}\cdot\sqrt{\sin\frac{C}{2}}>\frac{2}{3}\cdot\sqrt{3}\cdot\sqrt[4]{8}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a\sin\frac{A}{2}=\sum_{cyc}\frac{a\left(2\cos\frac{B+c}{2}\cos\frac{B-c}{2}\right)}{2\cos\frac{B-c}{2}} \overset{0<\cos\frac{B-c}{2}\leq 1 \text{ and analogs}}{\geq}$$

$$\sum_{cyc}\frac{a\left(2\cos\frac{B+c}{2}\cos\frac{B-c}{2}\right)}{2}=\sum_{cyc}\frac{a(\cos B+\cos C)}{2}=\frac{1}{2}\sum_{cyc}\left(a\left(\sum_{cyc}\cos A-\cos A\right)\right)$$

$$=\frac{1}{2}\left(\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}a\right)-\sum_{cyc}a\cos A\right)=\frac{1}{2}\left(\frac{2s(R+r)}{R}-R\sum_{cyc}\sin 2A\right)$$

$$=\frac{1}{2}\left(\frac{2s(R+r)}{R}-4R\prod_{cyc}\sin A\right)=\frac{1}{2}\left(\frac{2s(R+r)}{R}-\frac{4R.4Rrs}{8R^3}\right)$$

$$=\frac{1}{2}\left(\frac{2s(R+r)}{R}-\frac{2rs}{R}\right)=\frac{2Rs}{2R}=s \therefore \sum_{cyc}a\sin\frac{A}{2}\geq s\rightarrow(1)$$

$$\text{Now, }\frac{a}{h_a}\cdot\sqrt{\sin\frac{A}{2}}+\frac{b}{h_b}\cdot\sqrt{\sin\frac{B}{2}}+\frac{c}{h_c}\cdot\sqrt{\sin\frac{C}{2}}=\frac{1}{2rs}\cdot\sum_{cyc}\left(a^2\cdot\sqrt{\sin\frac{A}{2}}\right)$$

$$=\frac{1}{2rs}\cdot\sum_{cyc}\left(\frac{a^2.\sin\frac{A}{2}}{\sqrt{\sin\frac{A}{2}}}\right)\overset{0<\sin\frac{A}{2}\leq\frac{1}{\sqrt{2}} \text{ and analogs as } \Delta \text{ ABC is acute}}{\geq}\frac{1}{2rs}\cdot\sum_{cyc}\left(\frac{a^2.\sin\frac{A}{2}}{\frac{1}{\sqrt[4]{2}}}\right)$$

$$\overset{\text{Chebyshev}}{\geq}\frac{\sqrt[4]{2}}{6rs}\cdot\left(\sum_{cyc}a\sin\frac{A}{2}\right)\left(\sum_{cyc}a\right)$$

$$\left(\begin{array}{c}\because \text{WLOG } a\text{ssuming } a \geq b \geq c \Rightarrow \sin\frac{A}{2} \geq \sin\frac{B}{2} \geq \sin\frac{C}{2} \\ \Rightarrow a\sin\frac{A}{2} \geq b\sin\frac{B}{2} \geq c\sin\frac{C}{2}\end{array}\right) \overset{\text{via }(1)}{\geq} \frac{\sqrt[4]{2}}{6rs}.(s)(2s)$$

$$= \frac{\sqrt[4]{2}.s}{3r} \overset{\text{Mitrinovic}}{\geq} \frac{\sqrt[4]{2}.3\sqrt{3}.r}{3r} = \sqrt[4]{2}.\sqrt{3} \overset{?}{>} \frac{2}{3}.\sqrt{3}.\sqrt[4]{8} \Leftrightarrow 3 \overset{?}{>} 2\sqrt{2} \Leftrightarrow 9 \overset{?}{>} 8 \rightarrow \text{true}$$

$$\therefore \frac{a}{h_a}.\sqrt{\sin\frac{A}{2}} + \frac{b}{h_b}.\sqrt{\sin\frac{B}{2}} + \frac{c}{h_c}.\sqrt{\sin\frac{C}{2}} > \frac{2}{3}.\sqrt{3}.\sqrt[4]{8}\ \forall \text{ acute } \Delta\, ABC \text{ (QED)}$$

1857. In ΔABC the following relationship holds:

$$\frac{h_a^n}{\sin\frac{A}{2}} + \frac{h_b^n}{\sin\frac{B}{2}} + \frac{h_c^n}{\sin\frac{C}{2}} \geq 2\,.\,3^{n+1}\,r^n, n \in N$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a \geq b \geq c\ then \sin\frac{A}{2} \geq \sin\frac{B}{2} \geq \sin\frac{C}{2}\ and$$

$$h_a \leq h_b \leq h_c\ and\ h_a^n \leq h_b^n \leq h_c^n, \frac{1}{\sin\frac{A}{2}} \leq \frac{1}{\sin\frac{B}{2}} \leq \frac{1}{\sin\frac{C}{2}}$$

$$\sum h_a^n = (2F)^n \sum \frac{1}{a^n} = (2rs)^n \sum \frac{(1)^{n+1}}{a^n} \overset{Radon}{\geq}$$

$$\geq \frac{(2rs)^n 3^{n+1}}{(a+b+c)^n} = \frac{(2rs)^n 3^{n+1}}{(2s)^n} = r^n 3^{n+1}\ (1)$$

$$\sum \frac{1}{\sin\frac{A}{2}} \overset{AM-GM}{\geq} 3\left(\frac{1}{\prod \sin\frac{A}{2}}\right)^{\frac{1}{3}} = 3\left(\frac{4R}{r}\right)^{\frac{1}{3}} \overset{Euler}{\geq} 6\ (2)$$

$$\frac{h_a^n}{\sin\frac{A}{2}} + \frac{h_b^n}{\sin\frac{B}{2}} + \frac{h_c^n}{\sin\frac{C}{2}} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum h_a^n\right)\left(\sum \frac{1}{\sin\frac{A}{2}}\right) \overset{(1)\&(2)}{\geq}$$

$$\geq \frac{1}{3} r^n 3^{n+1}.6 = 2\,.\,3^{n+1}\,r^n$$

$$Equality\ for\ a = b = c$$

1858. In ΔABC the following relationship holds:

$$\frac{a^n}{\sin\frac{A}{2}\sin\frac{B}{2}} + \frac{b^n}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{c^n}{\sin\frac{C}{2}\sin\frac{A}{2}} \geq 2^{n+2}.3^{\frac{(n+2)}{2}}.r^n, \quad \forall n \in N$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$abc = 4Rrs \overset{Euler\ \&\ Mitrinovic}{\geq} 4.2r.r.3\sqrt{3}r = \left(2r\sqrt{3}\right)^3\ (1)$$

$$and\ \prod \sin^2\frac{A}{2} = \left(\frac{r}{4R}\right)^2 \overset{Euler}{\leq} \frac{1}{64}\quad (2)$$

$$\frac{a^n}{\sin\frac{A}{2}\sin\frac{B}{2}} + \frac{b^n}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{c^n}{\sin\frac{C}{2}\sin\frac{A}{2}} \overset{Am-Gm}{\geq} \frac{3(abc)^{\frac{n}{3}}}{\left(\prod \sin^2\frac{A}{2}\right)^{\frac{1}{3}}} \overset{(1)\&(2)}{\geq}$$

$$\geq\ 3\cdot 2^n\cdot r^n\cdot 3^{\frac{n}{2}}\cdot 4 = 2^{n+2}\cdot 3^{\frac{(n+2)}{2}}\cdot r^n$$

$$Equality\ for\ a = b = c$$

1859. In ΔABC the following relationship holds:

$$\frac{\tan^5\frac{A}{2}}{\sin^3 A} + \frac{\tan^5\frac{B}{2}}{\sin^3 B} + \frac{\tan^5\frac{C}{2}}{\sin^3 C} \geq \frac{8}{27}$$

Proposed by Zaza Mzvavanadze-Georgia

Solution by Tapas Das-India

$$\sum \sin A = \frac{s}{R} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}}{2}\ (1)\ and\ \frac{4R+r}{s} \overset{Doucet}{\geq} \sqrt{3}\ (2)$$

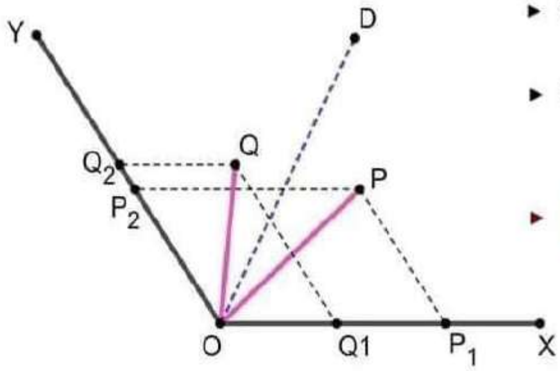
$$:\left(\frac{\tan^5\frac{A}{2}}{\sin^3 A} + \frac{\tan^5\frac{B}{2}}{\sin^3 B} + \frac{\tan^5\frac{C}{2}}{\sin^3 C}\right)\left(\sum \sin A\right)^3 (1+1+1) \overset{Holder}{\geq} \left(\sum \tan\frac{A}{2}\right)^5$$

$$or\left(\frac{\tan^5\frac{A}{2}}{\sin^3 A} + \frac{\tan^5\frac{B}{2}}{\sin^3 B} + \frac{\tan^5\frac{C}{2}}{\sin^3 C}\right) \geq \frac{\left(\sum \tan\frac{A}{2}\right)^5}{(\sum \sin A)^3(1+1+1)} = \frac{\left(\frac{4R+r}{s}\right)^5}{\left(3\left(\frac{s}{R}\right)^3\right)} \overset{(1)\&(2)}{\geq}$$

$$\geq \frac{3^{\frac{5}{2}}}{\frac{243\sqrt{3}}{8}} = \frac{9\cdot 8}{243} = \frac{8}{27}$$

$$Equality\ for\ A = B = C$$

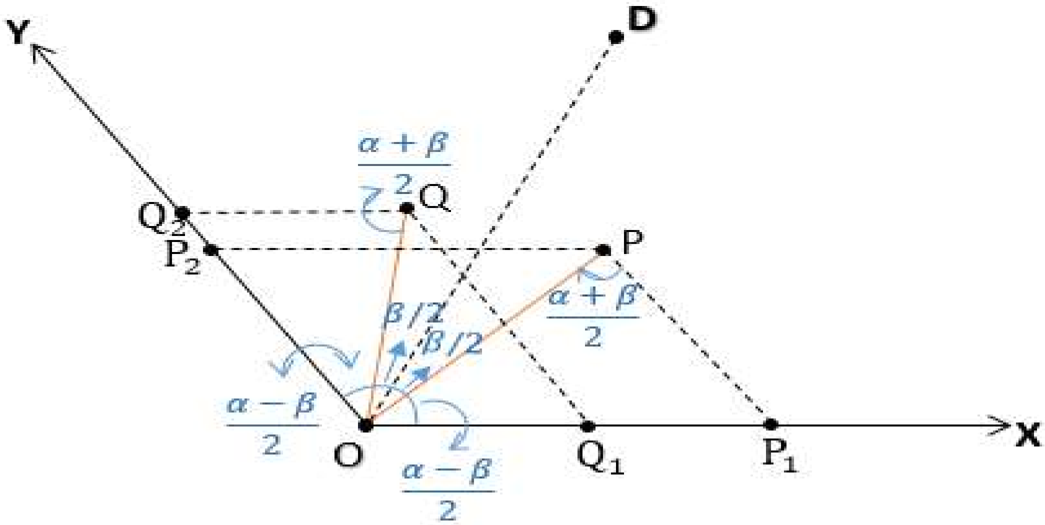
1860.



- OD bisector of $\angle XOY$ and $\angle POQ$. $\angle XOY = \theta$, $\angle POQ = \rho$
- PP_1OP_2 is #, QQ_1OQ_2 is #. $m_1 = \dfrac{OP_2}{OP_1}$, $m_2 = \dfrac{OQ_2}{OQ_1}$
- $m_1 = f_1(\theta, \rho) = ?$, $m_2 = f_2(\theta, \rho) = ?$

Proposed by Thanasis Gakopoulos – Greece

Solution by Mirsadix Muzefferov – Azerbaijan



$$m_1 = \frac{OP_2}{OP_1} = \frac{PP_1}{OP_1};\ in\ OPP_1\ rule\ sine:\ \frac{\sin\left(\frac{\theta-\rho}{2}\right)}{\sin\left(\frac{\theta+\rho}{2}\right)} = \frac{PP_1}{OP_1}$$

$$m_1 = \frac{OP_2}{OP_1} = \frac{PP_1}{OP_1} = \frac{\sin\left(\frac{\theta-\rho}{2}\right)}{\sin\left(\frac{\theta+\rho}{2}\right)} = \frac{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\rho}{2}\right) - \sin\left(\frac{\rho}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\rho}{2}\right) + \sin\left(\frac{\rho}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \frac{\tan\left(\frac{\theta}{2}\right) - \tan\left(\frac{\rho}{2}\right)}{\tan\left(\frac{\rho}{2}\right) + \tan\left(\frac{\theta}{2}\right)}$$

$$m_2 = \frac{OQ_2}{OQ_1} = \frac{OQ_2}{QQ_2};\ in\ OQQ_2\ rule\ sine:\ \frac{\sin\left(\frac{\theta+\rho}{2}\right)}{\sin\left(\frac{\theta-\rho}{2}\right)} = \frac{OQ_2}{QQ_2}$$

$$m_2=\frac{OQ_2}{OQ_1}=\frac{OQ_2}{QQ_2}=\frac{\sin\left(\frac{\theta+\rho}{2}\right)}{\sin\left(\frac{\theta-\rho}{2}\right)}=\frac{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\rho}{2}\right)+\sin\left(\frac{\rho}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\rho}{2}\right)-\sin\left(\frac{\rho}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$=\frac{\tan\left(\frac{\theta}{2}\right)+\tan\left(\frac{\rho}{2}\right)}{\tan\left(\frac{\theta}{2}\right)-\tan\left(\frac{\rho}{2}\right)}$$

1861. In ΔABC the following relationship holds:

$$\frac{1}{r_a(r_a+r_b)}+\frac{1}{r_b(r_b+r_c)}+\frac{1}{r_c(r_a+r_c)}\geq\frac{2}{3R^2}$$

Proposed by Ertan Yildirim-Turkiye

Solution by Tapas Das-India

$$(r_a+r_b)(r_b+r_c)(r_c+r_a)=$$
$$=\left(\sum r_a\right)\left(\sum r_ar_b\right)-r_ar_br_c=(4R+r)s^2-s^2r=4Rs^2$$

$$\frac{1}{r_a(r_a+r_b)}\cdot\frac{1}{r_b(r_b+r_c)}\cdot\frac{1}{r_c(r_a+r_c)}=$$

$$=\frac{1}{r_ar_br_c(r_a+r_b)(r_b+r_c)(r_c+r_a)}=\frac{1}{s^2r.4Rs^2}\overset{Euler\ \&\ Mitrinovic}{\geq}$$

$$\geq\frac{1}{\frac{27}{4}R^2.\frac{R}{2}.4.R.\frac{27}{4}R^2}=\frac{8}{R^6.3^6}\ (1)$$

$$,\frac{1}{r_a(r_a+r_b)}+\frac{1}{r_b(r_b+r_c)}+\frac{1}{r_c(r_a+r_c)}\overset{Am-Gm}{\geq}$$

$$\geq 3\sqrt[3]{\frac{1}{r_a(r_a+r_b)}\cdot\frac{1}{r_b(r_b+r_c)}\cdot\frac{1}{r_c(r_a+r_c)}}\overset{(1)}{\geq}3\sqrt[3]{\frac{8}{R^6.3^6}}=\frac{2}{3R^2}$$

$$Equality\ holds\ for\ a=b=c$$

1862. *In any ΔABC we have*:

$$\frac{a^2}{\sin^2 x}+\frac{b^2}{\cos^2 x}\geq\left(2\sqrt{11r^2+8Rr}-c\right)^2\ \forall x\in R-\left\{\frac{k\pi}{2}\right\}$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$s^2 \overset{Gerretsen}{\geq} 16Rr - 5r^2 = 8Rr + 8Rr - 5r^2 \overset{Euler}{\geq}$$

$$\geq 8.(2r).r + 8Rr - 5r^2 = 16r^2 + 8Rr - 5r^2 = 11r^2 + 8Rr \quad (1)$$

$$\frac{a^2}{\sin^2 x} + \frac{b^2}{\cos^2 x} \overset{Bergstrom}{\geq} \frac{(a+b)^2}{\cos^2 x + \sin^2 x} = (a+b)^2 = (2s-c)^2 =$$

$$= \left(2\sqrt{s^2} - c\right)^2 \overset{(1)}{\geq} \left(2\sqrt{11r^2 + 8Rr} - c\right)^2$$

1863. In ΔABC the following relationship holds:

$$\frac{1}{h_a(h_a+h_b)} + \frac{1}{h_b(h_b+h_c)} + \frac{1}{h_c(h_c+h_a)} \geq \frac{9}{2s^2}$$

Proposed by Ertan Yildirim-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{1}{h_a(h_a+h_b)} + \frac{1}{h_b(h_b+h_c)} + \frac{1}{h_c(h_c+h_a)} = \sum_{cyc} \frac{1}{h_a(h_a+h_b)} =$$

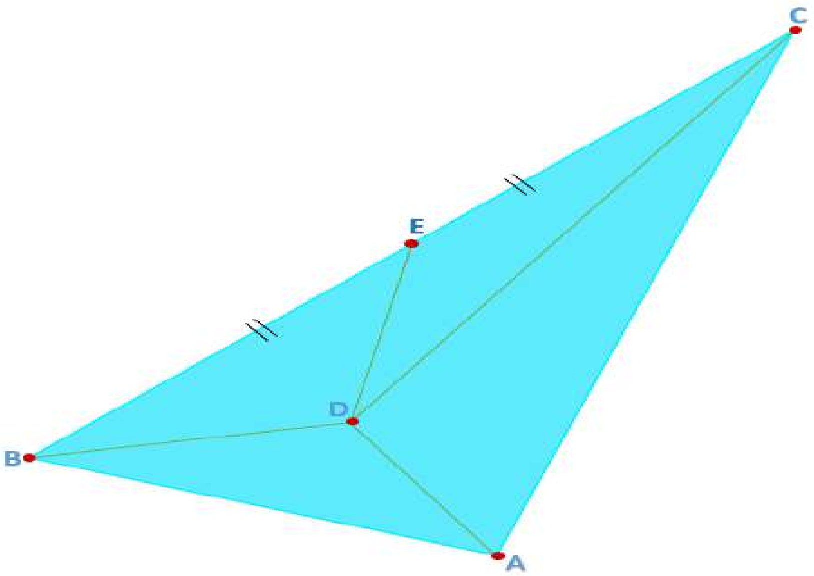
$$= \sum_{cyc} \frac{1}{\frac{2F}{a} \cdot \left(\frac{2F}{a} + \frac{2F}{b}\right)} = \frac{1}{4F^2} \cdot \sum_{cyc} \frac{1}{\frac{1}{a} \cdot \left(\frac{1}{a} + \frac{1}{b}\right)} = \frac{1}{4F^2} \cdot \sum_{cyc} \frac{a^2 b}{a+b} \geq$$

$$\overset{AM-GM}{\geq} \frac{1}{4F^2} \cdot 3 \cdot \sqrt[3]{\frac{a^3 b^3 c^3}{(a+b)(b+c)(c+a)}} =$$

$$= \frac{3abc}{4F^2} \cdot \frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}} \overset{AM-GM}{\geq} \frac{3 \cdot 4RF}{4F^2} \cdot \frac{1}{\frac{a+b+b+c+c+a}{3}} =$$

$$= \frac{9R}{F \cdot 4s} \overset{EULER}{\geq} \frac{18r}{4rs^2} = \frac{9}{2s^2}$$

Equality holds for: $a = b = c$.

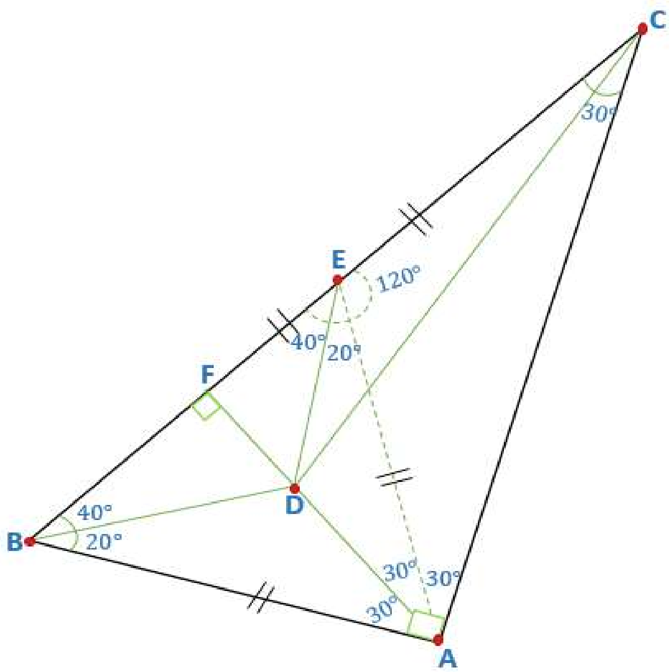


1864.

Suppose that $\angle DBA = 20^\circ$, $\angle DAB = 30^\circ$, $\angle DBC = 40^\circ$, $\angle DAC = 60^\circ$
Prove that : $\angle DEC = 140^\circ$

Proposed by Jafar Nikpour-Iran

Solution by Mirsadix Muzefferov-Azerbaijan

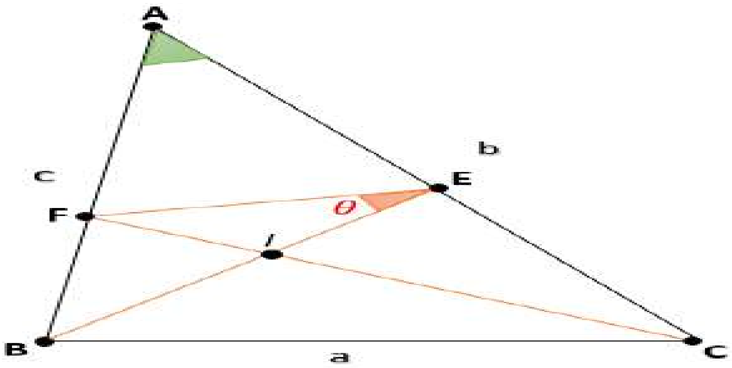


$\textit{Construct the media } AE. \Delta ABE \textit{ equilaterial triangle. Here } AE \textit{ bisectors}$
$D \in AF\, , C\hat{B}D = 40^\circ. \textit{Then } B\hat{E}D = 40^\circ.$

$$That's\ why\ \hat{AED} = 20°\ .On\ the\ other\ hand\ \hat{CEA} = 120°.$$

So ,

$$\hat{DEC} = \hat{AEC} + \hat{DEA} = 120° + 20° = 140°$$
$$\hat{DEC} = 140°$$



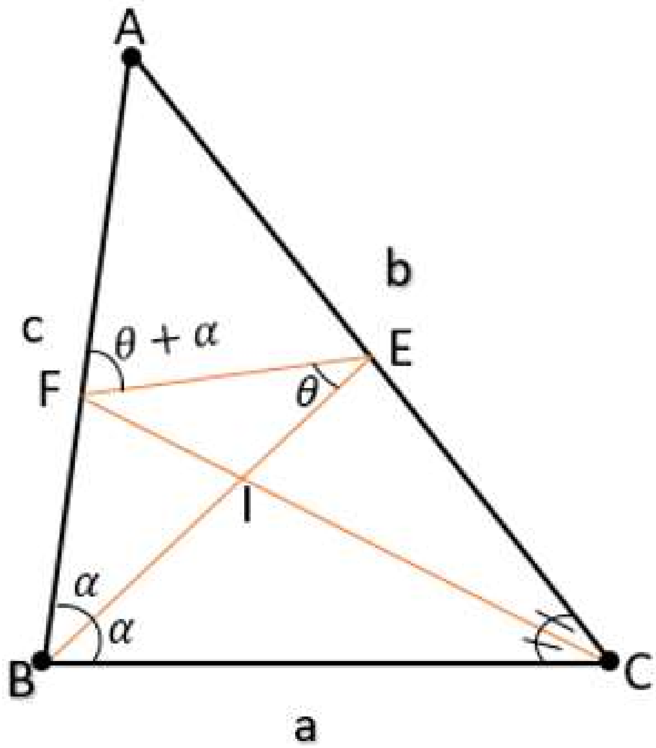
1865.

I-incenter.Prove that :

$$\cot\theta = \frac{1}{\sin A}\cdot\left[2-\cos A+\frac{b}{a+c}(1-2\cos A)\right]$$

Proposed by Thanasis Gakopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan



İn ΔABC BE and CF are the bisectors. Then:

$$AE=\frac{bc}{a+c}\ ;\ AF=\frac{bc}{a+b}\ ;BF=\frac{ac}{a+b}\quad (1)$$

İn ΔAFE rule cosine:

$$FE^2=AF^2+AE^2-2AF\cdot AE\cos A\ \ (2)$$

İn ΔABE rule cosine:

$$BE^2=AB^2+AE^2-2AB\cdot AE\cos A\ =$$
$$=(AF+FB)^2+AE^2-2(AF+FB)\cdot AE\cos A\ \ (2A)$$

İn ΔFEB rule sine :

$$\frac{\sin\theta}{BF}=\frac{\sin B\widehat{F}E}{BE}\Rightarrow BE=BF\cdot\frac{\sin(\theta+\alpha)}{\sin\theta}\ \ (3)$$

İn ΔAFE rule sine :

$$\frac{\sin(\theta+\alpha)}{AE}=\frac{\sin A}{FE}\Rightarrow FE=AE\cdot\frac{\sin A}{\sin(\theta+\alpha)}\ \ (4)$$

From (3) and (4) we have :

$$BE\cdot FE=BF\cdot AE\cdot\frac{\sin A}{\sin\theta}\ \ (5)$$

İn ΔFEB rule cosine :

$$\cos\theta=\frac{FE^2+BE^2-BF^2}{2FE\cdot BE}\ \ (*)$$

Let's use the expressions (2),(2A) and (5) in $(*)$

$$\cos\theta=\frac{AF^2+AE^2-2AF\cdot AE\cos A\ +(AF+FB)^2+AE^2-2(AF+FB)\cdot AE\cos A}{2BF\cdot AE\cdot\frac{\sin A}{\sin\theta}}$$

$$\cot(\theta)\sin(A)=\frac{AF^2}{BF\,AE}+\frac{AE}{BF}+\frac{AF}{AE}-\frac{2AF\cos(A)}{BF}-\cos(A)$$

Let's use expression (1)

$$\cot(\theta)\sin(A)=\left(\frac{bc}{a+b}\right)^2\frac{a+b}{ac}\frac{a+c}{b+c}+\frac{bc}{a+c}\frac{a+b}{ac}+\frac{bc}{a+b}\frac{a+c}{bc}$$
$$-\frac{2bc}{a+b}\frac{a+b}{ac}\cos(A)-\cos(A))$$
$$=\frac{(a+c)b}{(a+b)a}+\frac{b(a+b)}{a(a+c)}+\frac{a+c}{a+b}-\frac{2b}{a}\cos(A)-\cos(A))$$
$$=\frac{a+c}{a+b}\left(\frac{b}{a}+1\right)+\frac{b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)-\cos(A)$$
$$=\frac{a+c}{a}+\frac{b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)-\cos(A)$$
$$=\left((1-\cos(A))+\frac{c}{a}+\frac{b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)\right)$$
$$=1-\cos(A)+\frac{c(a+c)+b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)=$$
$$=1-\cos A+\frac{ac+c^2+ab+b^2}{a(a+c)}-\frac{2b}{a}\cdot\cos A=$$

$$= 1 - \cos(A) + \frac{ac + ab + a^2 + 2bc\cos(A)}{a(a+c)} - \frac{2b}{a}\cos(A)$$

$$= 1 - \cos(A) + 1 + \frac{b(a + 2c\cos(A))}{a(a+c)} - \frac{2b}{a}\cos(A)$$

$$= 2 - \cos(A) + \frac{b}{a+c}\left(\frac{a + 2\cos(A)}{a} - \frac{2(a+c)\cos(A)}{a}\right)$$

$$= 2 - \cos(A) + \frac{b}{a+c}\frac{a - 2a\cos(A)}{a} = 2 - \cos(A) + \frac{b}{a+c}(1 - 2\cos(A))$$

$$\cot(\theta) = \frac{1}{\sin(A)}\left(2 - \cos(A) + \frac{b}{a+c}(1 - 2\cos(A))\right)\ (proved)$$

1866. In ΔABC the following relationship holds:

$$(a+b+c)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \geq 6\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$(a+b+c)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) = 2s\left(\frac{s-a}{F} + \frac{s-b}{F} + \frac{s-c}{F}\right) =$$

$$= \frac{2s}{F}(3s - a - b - c) = \frac{2s}{rs}(3s - 2s) = \frac{2s}{r} \overset{MITRINOVIC}{\geq} \frac{2 \cdot 3\sqrt{3}r}{r} = 6\sqrt{3}$$

Equality holds for $a = b = c$.

1867. In ΔABC the following relationship holds:

$$(h_a + h_b + h_c)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \geq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$(h_a + h_b + h_c)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) = \sum_{cyc} h_a \cdot \sum_{cyc} \frac{1}{r_a} =$$

$$=\sum_{cyc}\frac{2F}{a}\cdot\sum_{cyc}\frac{s-a}{F}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\cdot(s-a+s-b+s-c)=$$

$$=2\cdot\frac{ab+bc+ca}{abc}\cdot(3s-a-b-c)=2\cdot\frac{s^2+r^2+4Rr}{4RF}\cdot(3s-2s)\overset{GERRETSEN}{\geq}$$

$$\geq 2\cdot\frac{16Rr-5r^2+r^2+4Rr}{4Rrs}\cdot s=2\cdot\frac{20Rr-4r^2}{4Rr}=$$

$$=10-\frac{2r}{R}\overset{EULER}{\geq}10-2\cdot\frac{R}{2}\cdot\frac{1}{R}=10-1=9$$

Equality holds for $a=b=c$.

1868. **In ΔABC the following relationship holds:**

$$\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}\leq\frac{\sqrt{3}R^2}{2r^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}=\sum_{cyc}\frac{a}{h_a}=\sum_{cyc}\frac{a}{\frac{2F}{a}}=\frac{1}{2F}\sum_{cyc}a^2=$$

$$=\frac{1}{2F}\cdot 2(s^2-r^2-4Rr)\overset{GERRETSEN}{\leq}\frac{1}{F}\cdot(4R^2+4Rr+3r^2-r^2-4Rr)=$$

$$=\frac{4R^2+2r^2}{rs}\overset{MITRINOVIC}{\leq}\frac{4R^2+2r^2}{r\cdot 3\sqrt{3}r}\overset{EULER}{\leq}\frac{4R^2+2\cdot\frac{R^2}{4}}{3\sqrt{3}r^2}=$$

$$=\frac{1}{3\sqrt{3}r^2}\cdot\frac{9R^2}{2}=\frac{\sqrt{3}R^2}{2r^2}$$

Equality holds for $a=b=c$.

1869. In ΔABC the following relationship holds:

$$\frac{(a+b)^3}{sin^2A}+\frac{(b+c)^3}{sin^2B}+\frac{(c+a)^3}{sin^2C}\geq 4\cdot\left(4\sqrt{3}r\right)^3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{(a+b)^3}{sin^2A}+\frac{(b+c)^3}{sin^2B}+\frac{(c+a)^3}{sin^2C}=\sum_{cyc}\frac{(a+b)^3}{sin^2A}=$$

$$=\sum_{cyc}\frac{(a+b)^3}{\left(\frac{a}{2R}\right)^2}=4R^2\sum_{cyc}\frac{(a+b)^3}{a^2}\overset{RADON}{\geq}$$

$$\geq 4R^2\cdot\frac{(a+b+b+c+c+a)^3}{(a+b+c)^2}=4R^2\cdot 8\cdot\frac{(a+b+c)^3}{(a+b+c)^2}=4R^2\cdot 8\cdot 2s\geq$$

$$\overset{EULER}{\geq} 4\cdot 4r^2\cdot 16\cdot s\overset{MITRINOVIC}{\geq} 4\cdot 4^3r^2\cdot 3\sqrt{3}r=4\cdot\left(4\sqrt{3}r\right)^3$$

Equality holds for $a=b=c$.

1870. If in ΔABC holds $\frac{\sum\frac{a}{r_a-r}}{\sin A+\sin B+\sin C}\leq 2$ then ΔABC is an equilateral one.

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$In\ an\ equilateral\ triangle\ holds\ R=2r$$

$$Proof: Let\ us\ assume\ \Delta ABC\ is\ equilateral\ and$$

$$AB=BC=CA=a\ and\ A=B=C=\frac{\pi}{3}$$

$$a=2R\sin A=2R\sin\frac{\pi}{3}=2R\frac{\sqrt{3}}{2}\ or, R=\frac{a}{\sqrt{3}}\ and$$

$$[ABC]=\sqrt{3}\frac{a^2}{4}, r=\frac{[ABC]}{\frac{a+a+a}{2}}=\frac{a}{2\sqrt{3}}$$

$$\frac{R}{r}=\frac{\frac{a}{\sqrt{3}}}{\frac{a}{2\sqrt{3}}}=2\ or\ R=2r\ \ (1)$$

$$\sum\frac{a}{r_a-r}=\sum\frac{a}{\frac{F}{s-a}-r}=\sum\frac{a}{r.\frac{s}{s-a}-r}=\sum\frac{a(s-a)}{rs-r(s-a)}=\frac{1}{r}\sum\frac{a(s-a)}{a}$$

$$=\frac{1}{r}\sum(s-a)=\frac{1}{r}(3s-2s)=\frac{s}{r}\ \ (2)$$

$$\sin A+\sin B+\sin C=\frac{s}{R}\ \ (3)$$

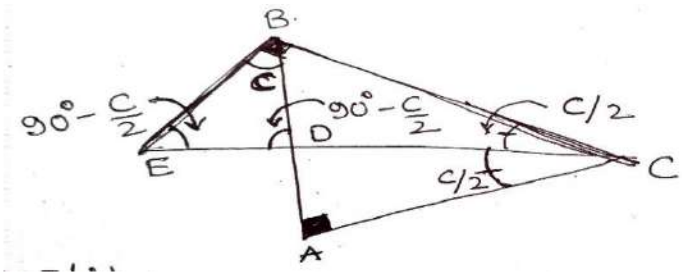
$$\frac{\sum\frac{a}{r_a-r}}{\sin A+\sin B+\sin C}\le 2\ or,\frac{\left(\frac{s}{r}\right)}{\frac{s}{R}}\le 2\ (using\ \ (3)\&\ (2))or$$

$$\frac{R}{r}\le 2\ ,we\ \ know\ that\frac{R}{r}\ge 2\ (Euler)$$

$$So\ \ we\ can\ \ say\ \frac{R}{r}=2\ ,and\ \ using\ (1)\ we\ \ can\ \ say\ \ \Delta ABC\ \ equilateral$$

1871.

$$In\ \Delta ABC\ \angle A=90^\circ, CD-internal\ bisector,$$
$$D\in(AB), BE\ perpendicular\ on\ BC,$$
$$CD\cap BE=\{E\}, R_1, R_2-Circumradii\ of\ \Delta BCE\ ,\Delta BDE.$$
$$Prove\ that:$$



$$2[BDE]<BD^2<\frac{(R_1+R_2)^2}{2}$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$Clearly,\angle EBC=\angle DAC=90^\circ,\angle BCE=\angle DCA=\frac{C}{2}and$$

$$\angle BED=\angle ADC=\angle BDE=90^\circ-\frac{C}{2}\ ,from\ \Delta EBD\ we\ \ have:$$

$$BE=BD(\ Since\ \ \angle BED=\angle BDE)\ (1)\ and\ \angle EBD=C$$

$$[BDE]=\frac{1}{2}\ BE.BD.\sin\angle EBD\overset{(1)}{=}\frac{1}{2}BD^2\sin C\ or$$

$$2[BDE]=BD^2\ \sin C<BD^2(as\ \ \sin C\le 1)\ \ (*)$$

$$from\ \Delta\ BDE\ ,2R_2=\frac{BD}{\sin\angle BED}=\frac{BD}{\sin\left(90^\circ-\frac{C}{2}\right)}=\frac{BD}{\cos\frac{C}{2}}\ or,2R_2\cos\frac{C}{2}=BD\ \ (2)$$

$$from\ \Delta BEC\ ,2R_1=\frac{BE}{\sin\frac{C}{2}}\overset{(1)}{=}\frac{BD}{\sin\frac{C}{2}}\ or,2R_1\sin\frac{C}{2}=BD\ \ (3)$$

$$Now\ from\ (2)\&\ (3)\ we\ get\ BD^2=\ BD.BD=\left(2R_2\cos\frac{C}{2}\right)\left(2R_1\sin\frac{C}{2}\right)$$

$$=2R_1R_2\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)=2R_1R_2\sin C$$

$$<2R_1R_2(\ as\ \sin C<1)\overset{AM-GM}{<}\ 2\left(\frac{R_1+R_2}{2}\right)^2=\frac{(R_1+R_2)^2}{2}(**)$$

$$Now\ from\ (*)\&(**)we\ have\ 2[BDE]<BD^2<\frac{(R_1+R_2)^2}{2}$$

1872.

Let ABC be the triangle in which the following relationship holds:

$$\left(1-\frac{r_a}{r_b}\right)\left(1-\frac{r_a}{r_c}\right)=2$$

$$Prove\ \ that:\ \frac{1}{e^{sinB}}+\frac{1}{e^{sinC}}\geq\frac{2}{e^{\frac{1}{\sqrt{2}}}}$$

Proposed by Tapas Das-India

Solution by Mirsadix Muzefferov-Azerbaijan

$$\left(1-\frac{r_a}{r_b}\right)\left(1-\frac{r_a}{r_c}\right)=2\ \ =>$$

$$(r_b-r_a)(r_c-r_a)=2.r_b.r_c\ \ =>\ \ r_br_c-r_ar_b-r_ar_c+(r_a)^2=2.r_b.r_c$$

$$r_ar_b+r_br_c+r_ar_c=(r_a)^2=>\ \boxed{r_a=p\ .\ tg\frac{A}{2}}$$

$$tg\frac{A}{2}.tg\frac{B}{2}+\ tg\frac{B}{2}.tg\frac{C}{2}+tg\frac{A}{2}.tg\frac{C}{2}=tg^2\frac{A}{2}\ \to\ tg^2\frac{A}{2}=1\to$$

$$A=90^\circ\ (Right\ \ triangle)$$

$$Here:\ \boxed{tg\frac{A}{2}.tg\frac{B}{2}+\ tg\frac{B}{2}.tg\frac{C}{2}+tg\frac{A}{2}.tg\frac{C}{2}=1}\ (true)$$

$$\frac{1}{e^{sinB}}+\frac{1}{e^{sinC}}=e^{-sinB}+e^{-sinC}\overset{A-G}{\geq}\ 2\sqrt{e^{-sinB-sinC}}$$

$$sinB+sinC=2sin\frac{B+C}{2}.cos\frac{B-C}{2}=2cos\frac{A}{2}.cos\frac{B-C}{2}\overset{A=90^\circ}{=}\ 2.\frac{1}{\sqrt{2}}.cos\frac{B-C}{2}$$

$$\leq\frac{2}{\sqrt{2}}$$

$$\frac{1}{e^{sinB}}+\frac{1}{e^{sinC}}\geq2.\sqrt{e^{-\frac{2}{\sqrt{2}}}}=\frac{2}{e^{\frac{1}{\sqrt{2}}}}\ \ (Proved)$$

1873. In ΔABC the following relationship holds:

$$\frac{1}{-a+b+c}\sqrt{\tan A}+\frac{1}{a-b+c}\sqrt{\tan B}+\frac{1}{a+b-c}\sqrt{\tan C}\geq\frac{1}{2r}\sqrt[4]{27}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Tapas Das-India

$$A+B+C=\pi \text{ or } A+B=\pi-A \text{ or } \tan(A+B)=-\tan C \text{ or}$$

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C \text{ or } \sum\tan A=\prod\tan A \quad (1)$$

$$\prod\tan A=\sum\tan A\overset{Jensen}{\geq}3\tan\frac{\pi}{3}=3\sqrt{3}=\left(\sqrt{3}\right)^3 \quad (2)$$

$$\sum\frac{1}{s-a}=\frac{3s^2-2s(a+b+c)+ab+bc+ca}{(s-a)(s-b)(s-c)}=$$

$$=\frac{r(4R+r)}{sr^2}=\frac{1}{r}\frac{4R+r}{s}\overset{Doucet}{\geq}\frac{\sqrt{3}}{r} \quad (3)$$

$$\frac{1}{-a+b+c}\sqrt{\tan A}+\frac{1}{a-b+c}\sqrt{\tan B}+\frac{1}{a+b-c}\sqrt{\tan C}=$$

$$=\frac{1}{2(s-a)}\sqrt{\tan A}+\frac{1}{2(s-b)}\sqrt{\tan B}+\frac{1}{2(s-c)}\sqrt{\tan C}\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{1}{2}\left(\sum\frac{1}{s-a}\right)\left(\sum\sqrt{\tan A}\right)\overset{AM-GM}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{1}{2}\left(\sum\frac{1}{s-a}\right).3\left(\prod\tan A\right)^{\frac{1}{6}}\overset{(3)\&(2)}{\geq}\frac{1}{6}\frac{\sqrt{3}}{r}3\left(\left(\sqrt{3}\right)^3\right)^{\frac{1}{6}}=$$

$$=\frac{\sqrt{3}}{2r}\left(\sqrt{3}\right)^{\frac{1}{2}}=\frac{\left(\sqrt{3}\right)^{\frac{3}{2}}}{2r}=\frac{3^{\frac{3}{4}}}{2r}=\frac{1}{2r}\sqrt[4]{27}$$

Equality holds for $a=b=c$.

1874. In ΔABC the following relationship holds:

$$\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\cdot\frac{b^2\sin\frac{B}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}+\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}\cdot\frac{a^2\sin\frac{A}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{A}{2}+a\sin\frac{C}{2}}+\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}\cdot\frac{a^2\sin\frac{A}{2}+b^2\sin\frac{B}{2}}{b\sin\frac{A}{2}+a\sin\frac{B}{2}}\ge 12\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\,, then.\sin\frac{A}{2}\ge\sin\frac{B}{2}\ge\sin\frac{C}{2}\ and\ \text{a}\sin\frac{A}{2}\ge\text{b}\sin\frac{B}{2}\ge c\sin\frac{C}{2}$$

$$\frac{b^2\sin\frac{B}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}=\frac{b\left(b\sin\frac{B}{2}\right)+c\left(c\sin\frac{C}{2}\right)}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}\overset{Chebyshev}{\ge}$$

$$\ge\frac{\frac{1}{2}(b+c)\left(b\sin\frac{B}{2}+c\sin\frac{C}{2}\right)}{\left(b\sin\frac{B}{2}+c\sin\frac{C}{2}\right)}=\frac{1}{2}(b+c)\overset{AM-GM}{\ge}\frac{2\sqrt{ab}}{2}=\sqrt{ab}$$

$$Similarly:\ \frac{a^2\sin\frac{A}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{A}{2}+a\sin\frac{C}{2}}\ge\sqrt{ac}\ and\ \frac{a^2\sin\frac{A}{2}+b^2\sin\frac{B}{2}}{b\sin\frac{A}{2}+a\sin\frac{B}{2}}\ge\sqrt{ab}$$

using above relation:

$$\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\cdot\frac{b^2\sin\frac{B}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}+\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}\cdot\frac{a^2\sin\frac{A}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{A}{2}+a\sin\frac{C}{2}}+\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}\cdot\frac{a^2\sin\frac{A}{2}+b^2\sin\frac{B}{2}}{b\sin\frac{A}{2}+a\sin\frac{B}{2}}\ge$$

$$\ge\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}.\sqrt{bc}+\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}.\sqrt{ac}+\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}.\sqrt{ab}\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}.\sqrt{bc}.\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}.\sqrt{ac}.\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}.\sqrt{ab}}=3\sqrt[3]{\frac{abc}{\prod\sin\frac{A}{2}}}$$

$$=3\left(4Rrs\cdot\frac{4R}{r}\right)^{\frac{1}{3}}\overset{Euler\ \&\ Mitrinovic}{\ge}3\left(4\cdot 2r\cdot r.3\sqrt{3}r\cdot 4\cdot 2r\right)^{\frac{1}{3}}=$$

$$= 3\left(64\, r^3(\sqrt{3}\,)^3\right)^{\frac{1}{3}} = 3.4.\sqrt{3}r = 12\sqrt{3}r$$

$$Equality\ for\ a = b = c$$

1875.

In any Δ ABC and $\forall$ n $\in \mathbb{N}$, the following relationship holds :

$$\sum_{cyc}\frac{b^{2n+1}\cot\frac{C}{2}+c^{2n+1}\cot\frac{B}{2}}{a(b^{n+1}+c^{n+1})}\ge 3\sqrt{3}(abc)^{\frac{n-1}{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle ($\because$ (A + B) + (B + C) > (C + A) and analogs) $\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \to (1)$$

$$\text{Now, } \forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \ge \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{b^{2n+1}\cot\frac{C}{2}+c^{2n+1}\cot\frac{B}{2}}{a(b^{n+1}+c^{n+1})}+\frac{c^{2n+1}\cot\frac{A}{2}+a^{2n+1}\cot\frac{C}{2}}{b(c^{n+1}+a^{n+1})}+\frac{a^{2n+1}\cot\frac{B}{2}+b^{2n+1}\cot\frac{A}{2}}{c(a^{n+1}+b^{n+1})}$$

$$=\frac{\frac{b^n}{c^{n+1}}\cot\frac{C}{2}+\frac{c^n}{b^{n+1}}\cot\frac{B}{2}}{a\left(\frac{1}{c^{n+1}}+\frac{1}{b^{n+1}}\right)}+\frac{\frac{c^n}{a^{n+1}}\cot\frac{A}{2}+\frac{a^n}{c^{n+1}}\cot\frac{C}{2}}{b\left(\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}\right)}+\frac{\frac{a^n}{b^{n+1}}\cot\frac{B}{2}+\frac{b^n}{a^{n+1}}\cot\frac{A}{2}}{c\left(\frac{1}{b^{n+1}}+\frac{1}{a^{n+1}}\right)}$$

$$=\frac{\frac{1}{a^{n+1}}}{\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}}\cdot\left(\frac{c^n a^n\cot\frac{B}{2}}{b^{n+1}}+\frac{a^n b^n\cot\frac{C}{2}}{c^{n+1}}\right)+\frac{\frac{1}{b^{n+1}}}{\frac{1}{c^{n+1}}+\frac{1}{a^{n+1}}}\cdot\left(\frac{a^n b^n\cot\frac{C}{2}}{c^{n+1}}+\frac{b^n c^n\cot\frac{A}{2}}{a^{n+1}}\right)$$

$$+\frac{\frac{1}{c^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}}\cdot\left(\frac{b^n c^n\cot\frac{A}{2}}{a^{n+1}}+\frac{c^n a^n\cot\frac{B}{2}}{b^{n+1}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{a^{n+1}},y=\frac{1}{b^{n+1}},z=\frac{1}{c^{n+1}},A=\frac{b^n c^n\cot\frac{A}{2}}{a^{n+1}},B=\frac{c^n a^n\cot\frac{B}{2}}{b^{n+1}},C=\frac{a^n b^n\cot\frac{C}{2}}{c^{n+1}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{b^n c^n\cot\frac{A}{2}}{a^{n+1}}.\frac{c^n a^n\cot\frac{B}{2}}{b^{n+1}}\right)}=\sqrt{\frac{3(abc)^n}{(abc)^{n+1}}.\sum_{cyc}\left(c^{2n+1}.\cot\frac{A}{2}\cot\frac{B}{2}\right)}$$

$$\overset{A-G}{\geq}3.\sqrt{\frac{1}{abc}.\sqrt[3]{(abc)^{2n+1}.\left(\prod_{cyc}\cot\frac{A}{2}\right)^2}}=3\sqrt{(abc)^{\frac{2n-2}{3}}.\sqrt[3]{\frac{s^6}{(r_a r_b r_c)^2}}}$$

$$=3(abc)^{\frac{n-1}{3}}.\sqrt[6]{\frac{s^6}{r^2s^4}}=3(abc)^{\frac{n-1}{3}}.\sqrt[3]{\frac{s}{r}}\overset{\text{Mitrinovic}}{\geq}=3(abc)^{\frac{n-1}{3}}.\sqrt{3}$$

$$\therefore\frac{b^{2n+1}\cot\frac{C}{2}+c^{2n+1}\cot\frac{B}{2}}{a(b^{n+1}+c^{n+1})}+\frac{c^{2n+1}\cot\frac{A}{2}+a^{2n+1}\cot\frac{C}{2}}{b(c^{n+1}+a^{n+1})}+\frac{a^{2n+1}\cot\frac{B}{2}+b^{2n+1}\cot\frac{A}{2}}{c(a^{n+1}+b^{n+1})}$$

$$\geq 3\sqrt{3}(abc)^{\frac{n-1}{3}}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1876.

In any $\Delta\,ABC$, the following relationship holds :

$$\frac{h_a+w_b}{m_a r_b}+\frac{h_b+w_c}{m_b r_c}+\frac{h_c+w_a}{m_c r_a}\geq\frac{8r}{R^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a+w_b}{m_a r_b}+\frac{h_b+w_c}{m_b r_c}+\frac{h_c+w_a}{m_c r_a}\ge$$

$$\ge \frac{h_a}{m_a r_b}+\frac{h_b}{m_b r_c}+\frac{h_c}{m_c r_a}+\frac{h_b}{m_a r_b}+\frac{h_c}{m_b r_c}+\frac{h_a}{m_c r_a}\overset{A-G}{\ge}$$

$$\ge 6\sqrt[6]{\frac{\left(\prod_{cyc}h_a\right)^2}{\left(\prod_{cyc}m_a\right)^2\left(\prod_{cyc}r_a\right)^2}}\overset{m_a m_b m_c\le\frac{Rs^2}{2}}{\ge} 6.\sqrt[3]{\frac{2r^2s^2}{R.\frac{Rs^2}{2}.rs^2}}\overset{\text{Mitrinovic}}{\ge}$$

$$\ge 6.\sqrt[3]{\frac{2r^2s^2}{R.\frac{Rs^2}{2}.r.\frac{27R^2}{4}}}=2\sqrt[3]{\frac{16r}{R^4}}\overset{?}{\ge}\frac{8r}{R^2}$$

$$\Leftrightarrow \frac{16r}{R^4}\overset{?}{\ge}\frac{64r^3}{R^6}\Leftrightarrow R^2\overset{?}{\ge}4r^2\to \text{true via Euler}\therefore \frac{h_a+w_b}{m_a r_b}+\frac{h_b+w_c}{m_b r_c}+\frac{h_c+w_a}{m_c r_a}$$

$$\ge\frac{8r}{R^2}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral}$$

$$\boxed{\text{Proof of } m_a m_b m_c\le\frac{Rs^2}{2}}$$

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\to(1)$$

$$\text{Now,}\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\to(2)$$

$$\text{Also}, \sum_{cyc} a^4 b^2 + \sum_{cyc} a^2 b^4 = \sum_{cyc}\left(a^2 b^2\left(\sum_{cyc} a^2 - c^2\right)\right) =$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \to (3) \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\begin{pmatrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{pmatrix}$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\begin{pmatrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{pmatrix}$$

$$= \frac{1}{16}(s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3)$$

$$\leq \frac{R^2 s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\leq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$
$$-r^3(4R+r)^3 \overset{?}{\leq} 0$$
$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \underbrace{\overset{?}{\geq}}_{(\bullet\bullet)} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(*)} s^2(16Rr - 5r^2)(8R - 16r)$$
$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \text{ and}$$
$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\leq}}_{(**)} 20rs^2(4R^2 + 4Rr + 3r^2)$$

$(*), (**) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$
$$\geq 20rs^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\geq 0$$
$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\geq}27r^2s^2$$

Now, LHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(***)}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\leq}}_{(****)}27r^2(4R^2+4Rr+3r^2)$

$(***), (****)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\geq 27r^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 224t^3-587t^2+308t-60\geq 0\ \left(\text{where } t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big)\geq 0\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$
$$\Rightarrow(\bullet)\text{ is true}\Rightarrow m_a^2m_b^2m_c^2\leq\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\leq\frac{Rs^2}{2}\ (\text{QED})$$

1877. In ΔABC the following relationship holds:

$$\sum\frac{\cot\frac{B}{2}\frac{\sin C}{\sin A}+\cot\frac{C}{2}\frac{\sin B}{\sin A}}{\sin C+\sin B}\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ \cot\frac{A}{2}\leq\cot\frac{B}{2}\leq\cot\frac{C}{2}\ and\ \sin A\geq\sin B\geq\sin C$$

$$\sum\frac{\cot\frac{B}{2}\frac{\sin C}{\sin A}+\cot\frac{C}{2}\frac{\sin B}{\sin A}}{\sin C+\sin B}\overset{Chebyshev}{\geq}\sum\frac{\frac{1}{\sin A}\left(\frac{1}{2}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)(\sin C+\sin B)\right)}{\sin C+\sin B}=$$

$$=\frac{1}{2}\sum\frac{1}{\sin A}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\overset{AM-GM}{\geq}\frac{1}{2}\sum\frac{1}{\sin A}2\sqrt{\left(\cot\frac{B}{2}\cot\frac{C}{2}\right)}=$$

$$=\sum\frac{1}{\sin A}\sqrt{\left(\cot\frac{B}{2}\cot\frac{C}{2}\right)}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\prod\cot\frac{A}{2}}{\prod\sin A}}=3\sqrt[3]{\frac{\frac{s}{r}}{\frac{sr}{2R^2}}}=3\sqrt[3]{2\left(\frac{R}{r}\right)^2}\overset{Euler}{\geq}3\sqrt[3]{8}=6$$

$$Equality\ holds\ for\ A=B=C$$

1878. In ΔABC the following relationship holds:

$$\frac{(a+b)^3}{m_a+r_b}+\frac{(b+c)^3}{m_b+r_c}+\frac{(c+a)^3}{m_c+r_a}\ge\frac{192\sqrt{3}r^3}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{(a+b)^3}{m_a+r_b}+\frac{(b+c)^3}{m_b+r_c}+\frac{(c+a)^3}{m_c+r_a}\overset{Holder}{\ge}\frac{(2a+2b+2c)^3}{3(\sum m_a+\sum r_a)}\overset{Leuenberger}{\ge}$$

$$\ge\frac{64s^3}{3(4R+r).2}\overset{Mitrinovic\ \&\ Euler}{\ge}64\cdot 27r^2\cdot\frac{3\sqrt{3}r}{6\frac{9R}{2}}=\frac{192\sqrt{3}r^3}{R}$$

$$Equality\ holds\ for\ a=b=c$$

1879. In ΔABC the following relationship holds:

$$\sum\frac{\cot^n\frac{B}{2}\frac{\sin C}{\sin A}+\cot^n\frac{C}{2}\frac{\sin B}{\sin A}}{\sin C+\sin B}\ge 6\cdot 3^{\frac{n-1}{2}},n\in\mathbb{N}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \cot\frac{A}{2}\le\cot\frac{B}{2}\le\cot\frac{C}{2}\ and\ \sin A\ge\sin B\ge\sin C$$

$$\sum\frac{\cot^n\frac{B}{2}\frac{\sin C}{\sin A}+\cot^n\frac{C}{2}\frac{\sin B}{\sin A}}{\sin C+\sin B}\overset{Chebycv}{\ge}\sum\frac{\frac{1}{\sin A}\left(\frac{1}{2}\left(\cot^n\frac{B}{2}+\cot^n\frac{C}{2}\right)(\sin C+\sin B)\right)}{\sin C+\sin B}=$$

$$=\frac{1}{2}\sum\frac{1}{\sin A}\left(\cot^n\frac{B}{2}+\cot^n\frac{C}{2}\right)\overset{AM-GM}{\ge}\frac{1}{2}\sum\frac{1}{\sin A}2\sqrt{\left(\cot^n\frac{B}{2}\cot^n\frac{C}{2}\right)}=$$

$$=\sum\frac{1}{\sin A}\sqrt{\left(\cot^n\frac{B}{2}\cot^n\frac{C}{2}\right)}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\prod\cot^n\frac{A}{2}}{\prod\sin A}}=3\sqrt[3]{\frac{\left(\frac{s}{r}\right)^n}{\frac{sr}{2R^2}}}=3\sqrt[3]{2\left(\frac{R}{r}\right)^2\left(\frac{s}{r}\right)^{n-1}}\overset{Euler}{\geq}$$

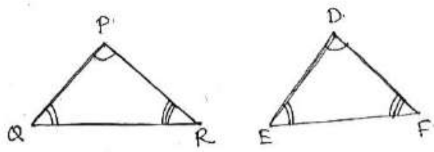
$$3\sqrt[3]{8}\left(\frac{s}{r}\right)^{\frac{n-1}{3}}\overset{Mitrinovic}{\geq}6\left(\sqrt{3}\right)^{n-1}=6\cdot 3^{\frac{n-1}{2}}$$

Equality holds for $A=B=C$

1880.

ABC, $A'B'C'$ and $A"B"C"$ are 3 similar triangles. If the sides $BC, B'C'$ and $B"C"$ are sides in a right triangle then the area of the biggest triangle is the sum of the areas of the other two triangle.

Proposed by Ioannis Stampouloglou-Greece



Solution by Tapas Das-India

Theorem: If ΔPQR and ΔDEF are similar $(\angle P=\angle D, \angle Q=\angle E, \angle R=\angle F)$ then

$$\frac{|PQR]}{[DEF]}=\frac{QR^2}{EF^2}=\frac{PR^2}{DF^2}=\frac{PQ^2}{DE^2}$$

Proof: $\Delta PQR\sim\Delta DEF$ and $(\angle P=\angle D, \angle Q=\angle E, \angle R=\angle F)$ so corresponding sides are proportional

$$\frac{QR}{EF}=\frac{PR}{DF}=\frac{PQ}{DE}\ (1)$$

$$Now\ \frac{|PQR]}{[DEF]}=\frac{\frac{1}{2}PQ.PR\sin P}{\frac{1}{2}DE.DF.\sin D}(since\ \angle P=\angle D)=\frac{PQ}{DE}\cdot\frac{PR}{DF}\overset{(1)}{=}\frac{QR}{EF}\cdot\frac{QR}{EF}=\frac{QR^2}{EF^2}$$

$$Similarly\frac{|PQR]}{[DEF]}=\frac{PR^2}{DF^2}\ and\ \frac{|PQR]}{[DEF]}=\frac{PQ^2}{DE^2}\ or$$

$$\frac{|PQR]}{[DEF]}=\frac{QR^2}{EF^2}=\frac{PR^2}{DF^2}=\frac{PQ^2}{DE^2}(proof\ complete)$$

According to the given problem $\Delta ABC, \Delta A'B'C'$ and $\Delta A"B"C"$ are similar, let side $BC > BC$, $BC > B'C'$ and according to the question they form right angle triangle, so

$B"C"^2=BC^2+B'C'^2$ (2) *and we need to show* $[ABC]+[A'B'C']=[A"B"C"]$ *now using the above theorem we have*

$$\frac{[ABC]}{BC^2}=\frac{[A'B'C']}{B'C'^2}=\frac{[A"B"C"]}{B"C"^2}=K(say).$$

$$[ABC]=K.BC^2,[A'B'C']=K.B'C'^2,[A"B"C"]=K.B"C"^2$$

$$[ABC]+[A'B'C']=K.BC^2+K.B'C'^2=K(BC^2+B'C'^2)\overset{(2)}{=}KB"C"^2=[A"B"C"]$$

$$or\ [ABC]+[A'B'C']=[A"B"C"](proof\ complete)$$

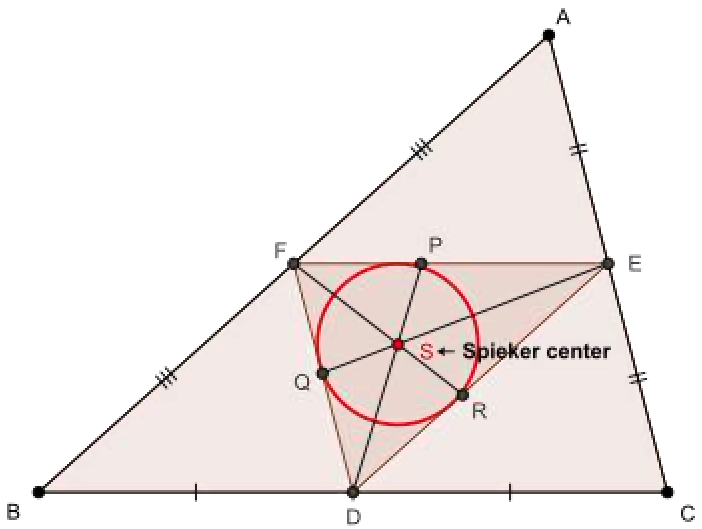
1881.

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left((2n_a+n_b).\sqrt{\frac{n_a}{n_b}}\right)\ge\sum_{cyc}\left(3p_a.\sqrt{\frac{w_a}{g_a}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and m(∡BAX) = α and m(∡CAX) = β (say) and inradius of Δ DEF = r′(say)

$$\text{Now, }16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because\text{ Spieker center is incenter of }\Delta\text{ DEF},\therefore m(∡AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and }m(∡AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18} \overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq} 0 \to \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2+\frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow \boxed{\sqrt{m_a n_a} \overset{(\blacksquare)}{\geq} p_a}$$

Now, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) \overset{(4)}{=} an_a^2+a(s-b)(s-c)$
$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$
$=an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2 \Rightarrow an_a^2=as^2+$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 \overset{(*****)}{=} s\left(s-a+\frac{(b-c)^2}{a}\right)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+$
$a(s-b)(s-c)$ and $b^2(s-b)+c^2(s-c) \overset{(5)}{=} ag_a^2+a(s-b)(s-c)$ and adding (4) and (5), we get:

$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$
$$\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+a^2-(b-c)^2$$
$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2) \Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)$$
$$\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 \overset{(******)}{=} (b-c)^2+2s(s-a)$$

Using (*****) and (******), we get : $g_a^2=(b-c)^2+2s(s-a)-s^2$

$$+\frac{4s(s-b)(s-c)}{a}=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}=(s-a)^2-$$

$$4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)$$

$$=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}=(s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow g_a^2 \overset{(*******)}{=} (s-a)\left(s-\frac{(b-c)^2}{a}\right) \therefore (*****),(*******) \Rightarrow n_a^2 g_a^2=$$

$$s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$

$$=s(s-a)\left(s(s-a)+s\frac{(b-c)^2}{a}-\frac{(b-c)^2}{a}(s-a)-\frac{(b-c)^4}{a^2}\right)$$

$$\Rightarrow n_a^2 g_a^2 \overset{(a)}{=} s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}\right)$$

$$\text{Again, } m_a^2 w_a^2 = \frac{(b-c)^2+4s(s-a)}{4}\cdot\frac{4bcs(s-a)}{(b+c)^2}$$

$$\Rightarrow m_a^2 w_a^2 \overset{(b)}{=} s(s-a)\frac{bc}{(b+c)^2}\left((b-c)^2+4s(s-a)\right) \therefore (1),(2) \Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2$$

$$= s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}-\frac{bc}{(b+c)^2}\left((b-c)^2+4s(s-a)\right)\right)$$

$$= s(s-a)\begin{pmatrix} s(s-a)+(b-c)^2\left(\dfrac{a^2-(b-c)^2}{a^2}\right) \\ -\dfrac{bc}{(b+c)^2}\left((b-c)^2+(b+c)^2-a^2\right)\end{pmatrix}$$

$$= s(s-a)\left(s(s-a)-bc+(a^2-(b-c)^2)\left(\frac{(b-c)^2}{a^2}+\frac{bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(((b+c)^2-a^2-4bc)+(a^2-(b-c)^2)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left((b-c)^2-a^2+(a^2-(b-c)^2)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}(a^2-(b-c)^2)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}-1\right)$$

$$= \frac{s(s-a)}{4}.4(s-b)(s-c)\left(\frac{4(b-c)^2}{a^2}-\frac{(b-c)^2}{(b+c)^2}\right) = r^2s^2(b-c)^2\left(\frac{4}{a^2}-\frac{1}{(b+c)^2}\right)$$

$$= r^2s^2(b-c)^2\left(\frac{2}{a}+\frac{1}{b+c}\right)\left(\frac{2b+2c-a}{a(b+c)}\right) \geq 0 \Rightarrow n_a^2 g_a^2 \geq m_a^2 w_a^2$$

$$\Rightarrow \boxed{\sqrt{n_a g_a} \overset{(\blacksquare\blacksquare)}{\geq} \sqrt{m_a w_a}}$$

$$\text{We have: } (2n_a+n_b).\sqrt{\frac{n_a}{n_b}} \overset{G-H}{\geq} (2n_a+n_b).\frac{2.\frac{n_a}{n_b}.1}{\frac{n_a}{n_b}+1} = (n_a+n_a+n_b).\frac{2n_a}{n_a+n_b}$$

$$= \frac{2n_a^2}{n_a+n_b}+2n_a \Rightarrow \sum_{cyc}\left((2n_a+n_b).\sqrt{\frac{n_a}{n_b}}\right) \geq 2\sum_{cyc}\frac{n_a^2}{n_a+n_b}+2\sum_{cyc}n_a$$

$$\overset{\text{Bergstrom}}{\geq} \frac{2\left(\sum_{cyc}n_a\right)^2}{2\sum_{cyc}n_a}+2\sum_{cyc}n_a \Rightarrow \boxed{\sum_{cyc}\left((2n_a+n_b).\sqrt{\frac{n_a}{n_b}}\right) \overset{(\blacksquare\blacksquare\blacksquare)}{\geq} 3\sum_{cyc}n_a}$$

$$\text{Now, }\frac{n_a}{p_a\cdot\sqrt{\frac{w_a}{g_a}}}=\frac{\sqrt{n_a g_a}\cdot\sqrt{\frac{n_a}{w_a}}}{p_a}\overset{\text{via }(\blacksquare\blacksquare)}{\geq}\frac{\sqrt{m_a w_a}\cdot\sqrt{\frac{n_a}{w_a}}}{p_a}=\frac{\sqrt{m_a n_a}}{p_a}\overset{\text{via }(\blacksquare)}{\geq}1$$

$$\Rightarrow\boxed{3n_a\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{\geq}3p_a\cdot\sqrt{\frac{w_a}{g_a}}}\Rightarrow\sum_{cyc}\left((2n_a+n_b)\cdot\sqrt{\frac{n_a}{n_b}}\right)\overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{\geq}3\sum_{cyc}n_a$$

$$\overset{\text{via }(\blacksquare\blacksquare\blacksquare\blacksquare)}{\geq}\sum_{cyc}\left(3p_a\cdot\sqrt{\frac{w_a}{g_a}}\right)\therefore\sum_{cyc}\left((2n_a+n_b)\cdot\sqrt{\frac{n_a}{n_b}}\right)\geq\sum_{cyc}\left(3p_a\cdot\sqrt{\frac{w_a}{g_a}}\right)$$

$$\forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1882.

In any Δ ABC with ω → Brocard's angle, the following relationship holds :

$$\sum_{cyc}\sqrt[4]{\left(2+\frac{1}{\sin\omega}\right)\left(3+\frac{n_a}{h_a}\right)}\geq\sum_{cyc}\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}a^2b^2}{4F^2}\overset{?}{\geq}\frac{(c^2+a^2)^2}{c^2a^2}\Leftrightarrow\frac{\sum_{cyc}a^4+16F^2}{8F^2}\overset{?}{\geq}\frac{c^4+a^4+2c^2a^2}{c^2a^2}$$

$$\Leftrightarrow\frac{\sum_{cyc}a^4}{8F^2}\overset{?}{\geq}\frac{c^4+a^4}{c^2a^2}\Leftrightarrow\frac{\sum_{cyc}a^4}{c^4+a^4}\overset{?}{\geq}\frac{8F^2}{c^2a^2}$$

$$\Leftrightarrow\frac{\sum_{cyc}a^4-(c^4+a^4)}{c^4+a^4}\overset{?}{\geq}\frac{2a^2b^2+2b^2c^2-\left(\sum_{cyc}a^4\right)}{2c^2a^2}$$

$$\Leftrightarrow\frac{b^4}{c^4+a^4}+\frac{b^4}{2c^2a^2}+\frac{c^4+a^4}{2c^2a^2}\overset{?}{\geq}\frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow\frac{b^4(c^2+a^2)^2}{2c^2a^2(c^4+a^4)}+\frac{c^4+a^4}{2c^2a^2}\overset{?}{\geq}\frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow b^4(c^2+a^2)^2+(c^4+a^4)^2\overset{?}{\geq}2b^2(c^2+a^2)(c^4+a^4)\to\text{true via AM}-\text{GM}$$

$$\therefore\frac{\sum_{cyc}a^2b^2}{4F^2}\geq\frac{(c^2+a^2)^2}{c^2a^2}\Rightarrow\frac{\sqrt{\sum_{cyc}a^2b^2}}{2F}\geq\frac{c}{a}+\frac{a}{c}\Rightarrow\frac{1}{\sin\omega}\overset{(*)}{\geq}\frac{c}{a}+\frac{a}{c}\text{ and analogs}$$

$$\text{Now, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s(a^2-(b-c)^2)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2\to(1)$$

$$\text{Now, } n_a\overset{?}{\geq}\frac{b^2-bc+c^2}{2R}\Leftrightarrow\frac{n_a}{h_a}\overset{?}{\geq}\frac{b^2-bc+c^2}{bc}\Leftrightarrow\frac{n_a^2}{h_a^2}-1\overset{?}{\geq}\left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$
$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\geq}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}$$
$$\overset{\text{via }(1)}{\Leftrightarrow}s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}\overset{?}{\geq}$$
$$\frac{(b-c)^2(b^2+c^2)}{b^2c^2}.\frac{b^2c^2}{4R^2}\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\geq}\frac{(b-c)^2(b^2+c^2)}{4R^2}\Leftrightarrow$$
$$\frac{s^2}{a^2}\overset{?}{\geq}\frac{b^2+c^2}{4R^2}\ (\because (b-c)^2\geq 0)\Leftrightarrow 4R^2s^2\overset{?}{\geq}a^2b^2+c^2a^2\to\text{true (strict inequality)}$$
$$\because 4R^2s^2\overset{\text{Goldstone}}{\geq}\sum_{cyc}a^2b^2>a^2b^2+c^2a^2\therefore n_a\overset{(**)}{\geq}\frac{b^2-bc+c^2}{2R}\text{ and analogs}$$

$$\therefore(*)\text{ and }(**)\Rightarrow\left(2+\frac{1}{\sin\omega}\right)\left(3+\frac{n_a}{h_a}\right)\geq\left(2+\frac{b}{c}+\frac{c}{b}\right)\left(3+\frac{b^2-bc+c^2}{2R.\frac{bc}{2R}}\right)$$
$$=\frac{(b+c)^2}{bc}.\frac{(b+c)^2}{bc}\Rightarrow\sqrt[4]{\left(2+\frac{1}{\sin\omega}\right)\left(3+\frac{n_a}{h_a}\right)}\geq\frac{b+c}{\sqrt{bc}}\text{ and analogs}$$
$$\therefore\sum_{cyc}\sqrt[4]{\left(2+\frac{1}{\sin\omega}\right)\left(3+\frac{n_a}{h_a}\right)}\geq\sum_{cyc}\sqrt{\frac{b}{c}}+\sum_{cyc}\sqrt{\frac{c}{b}}=\sum_{cyc}\sqrt{\frac{c}{a}}+\sum_{cyc}\sqrt{\frac{b}{a}}$$
$$\therefore\sum_{cyc}\sqrt[4]{\left(2+\frac{1}{\sin\omega}\right)\left(3+\frac{n_a}{h_a}\right)}\geq\sum_{cyc}\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\ \forall\,\Delta\,ABC,$$

with equality iff Δ ABC is equilateral (QED)

1883. In ΔABC the following relationship holds:

$$\frac{b+c-a}{m_a}+\frac{a+c-b}{m_b}+\frac{a+b-c}{m_c}\leq 2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Ertan Yildirim-Turkiye

Solution by Tapas Das-India

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\overset{C-S}{\geq}9\ \ (1)$$

$$\sum\sqrt{s-a}\overset{CBS}{\leq}\sqrt{3(3s-a-b-c)}=\sqrt{3s}\ \ (2)$$

$$\frac{b+c-a}{m_a}+\frac{a+c-b}{m_b}+\frac{a+b-c}{m_c}\overset{m_a\geq\sqrt{s(s-a)}}{\leq}$$

$$\frac{2(s-a)}{\sqrt{s(s-a)}}+\frac{2(s-b)}{\sqrt{s(s-b)}}+\frac{2(s-c)}{\sqrt{s(s-c)}}=\frac{2}{\sqrt{s}}\sum\sqrt{s-a}\overset{(2)}{\leq}\frac{2}{\sqrt{s}}\cdot\sqrt{3s}=2\sqrt{3}=$$

$$=2\cdot\frac{9}{3\sqrt{3}}\overset{(1)}{\leq}\frac{2}{3\sqrt{3}}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{4s}{3\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\overset{Mitrinovic}{\leq}$$

$$\leq\frac{\frac{4(3\sqrt{3}R)}{2}}{3\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Equality holds for $a=b=c$.

1884. In ΔABC the following relationship holds:

$$\frac{xr_a+yr_b}{zr_c}+\frac{yr_a+zr_c}{xr_b}+\frac{xr_c+zr_b}{yr_a}\geq 12\sqrt[3]{\frac{r}{4R}}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$\frac{xr_a+yr_b}{zr_c}+\frac{yr_a+zr_c}{xr_b}+\frac{xr_c+zr_b}{yr_a}=$$

$$=\left(\frac{x}{z}\frac{r_a}{r_c}+\frac{y}{x}\frac{r_a}{r_b}+\frac{x}{y}\frac{r_c}{r_a}\right)+\left(\frac{y}{z}\frac{r_b}{r_c}+\frac{z}{x}\frac{r_c}{r_b}+\frac{z}{y}\frac{r_b}{r_a}\right)\overset{AM-GM}{\geq}$$

$$\geq 6\left(\left(\frac{x}{z}\frac{r_a}{r_c}.\frac{y}{x}\frac{r_a}{r_b}.\frac{x}{y}\frac{r_c}{r_a}\right).\left(\frac{y}{z}\frac{r_b}{r_c}.\frac{z}{x}\frac{r_c}{r_b}.\frac{z}{y}\frac{r_b}{r_a}\right)\right)^{\frac{1}{3}}=$$

$$=6(1)^{\frac{1}{3}}=6=12\cdot\frac{1}{2}=12\left(\frac{1}{8}\right)^{\frac{1}{3}}=12\left(\frac{r}{8r}\right)^{\frac{1}{3}}\overset{Euler}{\geq}12\left(\frac{r}{\frac{8R}{2}}\right)^{\frac{1}{3}}=12\sqrt[3]{\frac{r}{4R}}$$

1885. In ΔABC the following relationship holds:

$$3\sqrt{3}r\leq\frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b}\leq\left(\frac{R}{r}-1\right)s$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ r_a\geq r_b\geq r_c\ and\ (r_a+r_b)\geq(r_a+r_c)\geq(r_b+r_c)$$
$$and\ \frac{r_a}{(r_b+r_c)}\geq\frac{r_b}{(r_a+r_c)}\geq\frac{r_c}{(r_a+r_b)}$$

$$\sum ar_a=F\sum\frac{a}{s-a}=\frac{F}{(s-a)(s-b)(s-c)}\left(\sum a(s-b)(s-c)\right)=$$

$$=\frac{F}{sr^2}(s^2(a+b+c)-2s(ab+bc+ca)+3abc)=$$

$$=\frac{F}{sr^2}(2s^3-2s(s^2+r^2+4Rr)+12Rrs)=\frac{F}{sr^2}(2Rr-r^2)2s=\frac{2s(2Rr-r^2)}{r}\quad(1)$$

$$\frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b}=\sum\frac{ar_a}{r_b+r_c}\overset{AM-HM}{\leq}\sum\frac{1}{4}a(\frac{r_a}{r_b}+\frac{r_a}{r_c})=$$

$$\sum\frac{1}{4}a\left(\frac{r_a}{r_b}+\frac{r_a}{r_c}+\frac{r_a}{r_a}-1\right)=\sum ar_a\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\frac{1}{4}-\sum\frac{a}{4}=$$

$$=\frac{1}{4r}\sum ar_a-\frac{s}{2}\overset{(1)}{=}\frac{2s(2Rr-r^2)}{r(4r)}-\frac{s}{2}=\frac{s}{2}\left(\frac{2R}{r}-2\right)=\left(\frac{R}{r}-1\right)s$$

$$\frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b}\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\left(\sum a\right)\left(\sum\frac{r_a}{r_b+r_c}\right)\overset{Nesbitt}{\geq}\frac{1}{3}2s.\frac{3}{2}\overset{Mitrinovic}{\geq}3\sqrt{3}r$$

Equality holds for $a=b=c$.

1886. In any Δ ABC, the following relationship holds :

$$\frac{9}{4s^2} \le \frac{1}{(r_a+h_a)^2}+\frac{1}{(r_b+h_b)^2}+\frac{1}{(r_c+h_c)^2} \le \frac{1}{12r^2}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a(s-a)}{b+c}=\sum_{cyc}\frac{(a-2s+2s)(s-a)}{2s-a}=$$

$$=-\sum_{cyc}(s-a)+2s^2\sum_{cyc}\frac{1}{b+c}+2s\sum_{cyc}\frac{-a+2s-2s}{2s-a}$$

$$=-s+2s^2\sum_{cyc}\frac{1}{b+c}+6s-4s^2\sum_{cyc}\frac{1}{b+c}=5s-2s^2.\sum_{cyc}\frac{(c+a)(a+b)}{\prod_{cyc}(b+c)}$$

$$=5s-2s^2.\frac{\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab}{2s(s^2+2Rr+r^2)}=5s-2s^2.\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}$$

$$=5s-s.\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}\Rightarrow\sum_{cyc}\frac{a(s-a)}{b+c}=\frac{s(6Rr+4r^2)}{s^2+2Rr+r^2}\to(1)$$

$$\text{Now, }\sum_{cyc}\frac{1}{(r_a+h_a)^2}=\sum_{cyc}\frac{1}{\left(\frac{rs}{s-a}+\frac{2rs}{a}\right)^2}=\frac{1}{r^2s^2}.\sum_{cyc}\frac{a^2(s-a)^2}{(b+c)^2}$$

$$\ge\frac{1}{3r^2s^2}.\left(\sum_{cyc}\frac{a(s-a)}{b+c}\right)^2\overset{\text{via }(1)}{\ge}\frac{s^2r^2(6R+4r)^2}{3r^2s^2(s^2+2Rr+r^2)^2}\overset{?}{\ge}\frac{9}{4s^2}$$

$$\Leftrightarrow 4s^2(6R+4r)^2\overset{?}{\ge}27(s^2+2Rr+r^2)^2$$

$$\Leftrightarrow(144R^2+84Rr+10r^2)s^2\underbrace{\overset{?}{\ge}}_{(*)}27s^4+27r^2(2R+r)^2$$

$$\text{Now, RHS of }(*)\overset{\text{Gerretsen}}{\le}(108R^2+108Rr+81r^2)s^2+27r^2(2R+r)^2$$

$$\overset{?}{\le}(144R^2+84Rr+10r^2)s^2\Leftrightarrow(36R^2-24Rr-71r^2)s^2\underbrace{\overset{?}{\ge}}_{(**)}27r^2(2R+r)^2$$

$$\text{Again, }\because 36R^2-24Rr-71r^2=(R-2r)(36R+48r)+25r^2\overset{\text{Euler}}{\ge}25r^2>0$$

$$\therefore\text{ LHS of }(**)\overset{\text{Gerretsen}}{\ge}(36R^2-24Rr-71r^2)(16Rr-5r^2)\overset{?}{\ge}27r^2(2R+r)^2$$

$$\Leftrightarrow 144t^3-168t^2-281t+82\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(144t^2+99t+21(t-2)+1)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\Rightarrow (**) \Rightarrow (*)\text{is true} \therefore \sum_{cyc}\frac{1}{(r_a+h_a)^2} \geq \frac{9}{4s^2}$$

$$\text{Also}, \sum_{cyc}\frac{1}{(r_a+h_a)^2} \overset{A-G}{\leq} \sum_{cyc}\frac{1}{4r_ah_a} = \sum_{cyc}\frac{a(s-a)}{8r^2s^2}$$

$$= \frac{1}{8r^2s^2}\cdot\left(s(2s)-2(s^2-4Rr-r^2)\right) = \frac{4R+r}{4rs^2} \overset{\text{Gerretsen + Euler}}{\leq} \frac{4R+r}{4r.3r(4R+r)}$$

$$\therefore \sum_{cyc}\frac{1}{(r_a+h_a)^2} \leq \frac{1}{12r^2} \text{ and so},$$

$$\frac{9}{4s^2} \leq \frac{1}{(r_a+h_a)^2}+\frac{1}{(r_b+h_b)^2}+\frac{1}{(r_c+h_c)^2} \leq \frac{1}{12r^2}\ \forall\Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1887. **If in ΔABC: $a = \frac{b+c}{2}$ then:**

$$\sin^2\frac{A}{2} \geq \frac{r}{2R}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Tapas Das-India

Lemma:

$$\cos\frac{B-C}{2} \geq \sqrt{\frac{2r}{R}}$$

(*Reference : Geometric Inequalities Marathon, First* 100 *problems and solutions*)(1)

$$\text{Mollweide's}: \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}} = \frac{b+c}{a}$$

$$\text{Now } a = \frac{b+c}{2} \text{ or } \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}} = \frac{b+c}{a} = 2$$

$$\text{or } 2\sin\frac{A}{2} = \cos\frac{B-C}{2} \overset{(1)}{\geq} \sqrt{\frac{2r}{R}} \text{ or } \sin^2\frac{A}{2} \geq \frac{r}{2R}$$

Solution 2 by Tapas Das-India

$$\cos\frac{B-C}{2} = \cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2} =$$

$$=\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}.+\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}=$$

$$=\frac{1}{a\sqrt{bc}}(2s-a)\sqrt{(s-b)(s-c)}=\frac{\frac{1}{a\sqrt{bc}}(b+c)\left(\sqrt{s(s-a)(s-b)(s-c)}\right)}{\sqrt{s(s-a)}}=$$

$$=\frac{\frac{1}{a\sqrt{bc}}(b+c)2\left(\sqrt{s(s-a)(s-b)(s-c)}\right)}{2\sqrt{s(s-a)}}=\frac{\frac{2F}{a}(b+c)}{2\sqrt{bcs(s-a)}}.=\frac{h_a}{w_a}=$$

$$=\frac{(b+c)bc}{2R.2\sqrt{bcs(s-a)}.}=\frac{(2(s-a)+a)\sqrt{bc}}{4R\sqrt{s(s-a)}}\overset{AM-GM}{\geq}$$

$$\geq\frac{2\sqrt{2(s-a)abc}}{4R\sqrt{s(s-a)}}=\frac{1}{2R}\sqrt{\frac{8Rsr}{s}}=\sqrt{\frac{2r}{R}}\ (1)$$

$$Mollweide's:\ \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\frac{b+c}{a}$$

$$Now\ a=\frac{b+c}{2}\ or\ \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\frac{b+c}{a}=2$$

$$or\ 2\sin\frac{A}{2}=\cos\frac{B-C}{2}\overset{(1)}{\geq}\sqrt{\frac{2r}{R}}\ or\sin^2\frac{A}{2}\geq\frac{r}{2R}$$

1888. In ΔABC the following relationship holds:

$$\frac{2}{R}\leq\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\leq\frac{R}{2r^2}$$

Proposed by Marian Ursărescu-Romania

Solution by Tapas Das-India

$$s_a=\frac{2bcm_a}{b^2+c^2}$$

$$m_a-s_a=m_a\left(1-2bc\frac{1}{b^2+c^2}\right)=\frac{m_a}{b^2+c^2}(b-c)^2\geq 0\ so\ m_a\geq\ s_a\ (1)$$

$$\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{s_a+s_b+s_c}\overset{(1)}{\geq}.\frac{9}{m_a+m_b+m_c}\overset{Leuenberger}{\geq}\frac{9}{4R+r}\overset{Euler}{\geq}\frac{9}{\frac{9R}{2}}=\frac{2}{R}$$

$$\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\leq\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}=\frac{r}{r^2}\overset{Euler}{\leq}\frac{\frac{R}{2}}{r^2}=\frac{R}{2r^2}$$

Equality for $a=b=c$.

1889. In any Δ ABC, the following relationship holds :

$$\frac{9}{8}\leq\sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right)\leq\frac{9R}{16r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right)=\frac{1}{2R}.\sum_{cyc}\frac{rsa}{(s-a)(2s-a)}$$

$$=\frac{r}{2R}.\sum_{cyc}\frac{a\big((2s-a)-(s-a)\big)}{(s-a)(2s-a)}=\frac{r}{2R}\left(\sum_{cyc}\frac{a-s+s}{s-a}-\sum_{cyc}\frac{a}{b+c}\right)$$

$$=\frac{r}{2R}\left(-3+\frac{s(4Rr+r^2)}{r^2s}-\sum_{cyc}\frac{a}{b+c}\right)\overset{(1)}{=}\frac{r}{2R}\left(\frac{4R-2r}{r}-\sum_{cyc}\frac{a}{b+c}\right)$$

$$=\frac{r}{2R}\left(\frac{4R-2r}{r}-\sum_{cyc}\frac{2s-(b+c)}{b+c}\right)=\frac{r}{2R}\left(\frac{4R-2r}{r}+3-2s.\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}\right)$$

$$=\frac{(4R-4r)s^2+2Rr(4R+r)}{2R(s^2+2Rr+r^2)}\overset{?}{\geq}\frac{9}{8}\Leftrightarrow(7R-16r)s^2+Rr(14R-r)\underbrace{\overset{?}{\geq}}_{(*)}0$$

$\boxed{Case\ 1}$ $7R-16r\geq 0$ *and* then : LHS of $(*)\geq Rr(14R-r)>0\Rightarrow(*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $7R-16r<0$ and then : LHS of $(*)\geq(7R-16r)(4R^2+4Rr+3r^2)$

$$+Rr(14R-r)\overset{?}{\geq}0\Leftrightarrow 14t^3-11t^2-22t-24\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(14t^2+17t+12)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{Euler}{\geq}2\Rightarrow(*)\text{ is true}$$

$\therefore$ combining both cases, $(*)$ is true for *all* triangles $\therefore\sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right)\geq\frac{9}{8}$

$$\text{Again, }\sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right)\overset{\text{via }(1)}{=}\frac{r}{2R}\left(\frac{4R-2r}{r}-\sum_{cyc}\frac{a}{b+c}\right)\overset{\text{Nesbitt}}{\le}\frac{r}{2R}\left(\frac{4R-2r}{r}-\frac{3}{2}\right)$$

$$=\frac{8R-7r}{4R}\overset{?}{\le}\frac{9R}{16r}\Leftrightarrow 9R^2-32Rr+28r^2\overset{?}{\ge}0\Leftrightarrow(9R-14r)(R-2r)\overset{?}{\ge}0\to\text{true}$$

$$\because R\overset{\text{Euler}}{\ge}2\Rightarrow\sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right)\le\frac{9R}{16r}\text{ and so, }\frac{9}{8}\ge\sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right)\le\frac{9R}{16r}$$

$$\forall\,\Delta\,ABC,\text{ with equality iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1890. In ΔABC the following relationship holds:

$$\frac{b}{s-b}+\frac{c}{s-c}\ge\frac{8R}{r}\cdot\frac{s-a}{a}\cdot\sin\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{\frac{b}{s-b}+\frac{c}{s-c}}{\frac{s-a}{a}\sin\frac{A}{2}}=\frac{1}{\sin\frac{A}{2}}\left(\frac{ab}{(s-b)(s-a)}+\frac{ac}{(s-c)(s-a)}\right)=$$

$$=\frac{1}{\sin\frac{A}{2}}\left(\frac{1}{\sin^2\frac{C}{2}}+\frac{1}{\sin^2\frac{B}{2}}\right)\overset{AM-GM}{\ge}\frac{2}{\sin\frac{A}{2}}\left(\frac{1}{\sin\frac{B}{2}}\frac{1}{\sin\frac{C}{2}}\right)=2.\left(\frac{4R}{r}\right)=\frac{8R}{r}$$

$$\text{or }\frac{b}{s-b}+\frac{c}{s-c}\ge\frac{8R}{r}\cdot\frac{s-a}{a}\cdot\sin\frac{A}{2}$$

$$\text{Equality for }a=b=c.$$

1891. Prove that:

$$\sin(6^\circ)=4\sin(12^\circ).\sin(18^\circ).\sin(24^\circ)$$

Proposed by Murat Oz-Turkiye

Solution by Shirvan Tahirov-Azerbaijan

$$\sin(\pi-x)=\sin x\to\sin\left(\pi-\frac{2\pi}{5}\right)=\sin\frac{2\pi}{5}$$

$$\sin\frac{3\pi}{5}=\sin\frac{2\pi}{5}\to\sin\frac{\pi}{5}\left(3-4\sin^2\frac{\pi}{5}\right)=2\sin\frac{\pi}{5}\cos\frac{\pi}{5}$$

$$3-4\sin^2\frac{\pi}{5}=2\cos\frac{\pi}{5}\to3-4\left(1-\cos^2\frac{\pi}{5}\right)=2\cos\frac{\pi}{5}$$

$$4cos^2\frac{\pi}{5}-2cos\frac{\pi}{5}-1=0\rightarrow cos\frac{\pi}{5}=\frac{\sqrt{5}+1}{4}$$

$$1-2sin^2\frac{\pi}{10}=\frac{\sqrt{5}+1}{4}\rightarrow sin^2\frac{\pi}{10}=\frac{3-\sqrt{5}}{8}\rightarrow sin^2\frac{\pi}{10}=\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$sin\frac{\pi}{10}=\frac{\sqrt{5}-1}{4}\rightarrow sin18^\circ=\frac{\sqrt{5}-1}{4}$$

We must prove that:

$$4sin12^\circ sin18^\circ sin24^\circ=sin6^\circ\Leftrightarrow$$
$$\Leftrightarrow 8cos6^\circ\,4sin12^\circ sin18^\circ sin24^\circ=2cos6^\circ sin6^\circ\Leftrightarrow$$
$$\Leftrightarrow 8cos6^\circ sin12^\circ sin18^\circ sin24^\circ=sin12^\circ\Leftrightarrow 8cos6^\circ sin18^\circ sin24^\circ=1\Leftrightarrow$$
$$\Leftrightarrow sin24^\circ cos6^\circ sin18^\circ=\frac{1}{8}\Leftrightarrow\frac{1}{2}\big(sin(24^\circ+6^\circ)+sin(24^\circ-6^\circ)\big)sin18^\circ=\frac{1}{8}\Leftrightarrow$$
$$\Leftrightarrow(sin30^\circ+sin18^\circ)sin18^\circ=\frac{1}{4}\Leftrightarrow\left(\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)\frac{\sqrt{5}-1}{4}=\frac{1}{4}\Leftrightarrow$$
$$\Leftrightarrow\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{5}-1}{4}=\frac{1}{4}\Leftrightarrow\frac{5-1}{16}=\frac{1}{4}\Leftrightarrow\frac{1}{4}=\frac{1}{4}$$

1892. In ΔABC the following relationship holds:

$$\sum\frac{h_a}{b+c}\sin A\ge\frac{9r}{4R}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{h_a}{b+c}\sin A=\sum\frac{bc}{2R}\cdot\frac{a}{2R}\cdot\frac{1}{b+c}=\frac{abc}{4R^2}\sum\frac{1}{b+c}\overset{Bergstrom}{\ge}$$

$$\ge\frac{abc}{4R^2}\cdot\frac{(1+1+1)^2}{2a+2b+2c}=\frac{abc}{4R^2}\cdot\frac{9}{4s}=4Rrs\cdot\frac{9}{16R^2s}=\frac{9r}{4R}$$

$$Equality\ holds\ for\ a=b=c$$

1893. In ΔABC the following relationship holds:

$$\sum h_a^3\le\sum r_a^3$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\textit{Lemma}: \ h_a^2 \le r_b r_c$$

$$\textit{Proof}: h_a^2 \le r_b r_c \text{ or } \frac{4F^2}{a^2} \le \frac{F^2}{(s-b)(s-c)} \text{ or } 4(s-b)(s-c) \le a^2$$

$$\text{or } 4(s-b)(s-c) \le \left((s-b)+(s-c)\right)^2 \text{ true}$$

$$\text{since } \left((s-b)+(s-c)\right)^2 \overset{AM-GM}{\ge} \left(2\left((s-b)(s-c)\right)^{\frac{1}{2}}\right)^2 = 4(s-b)(s-c)$$

$$\sum r_a^3 = \left(\sum r_a\right)^3 - 3\left(\left(\sum r_a\right)\left(\sum r_a r_b\right) - r_a r_b r_c\right) =$$
$$= (4R+r)^3 - 3\left((4R+r)s^2 - s^2 r\right) = (4R+r)^3 - 12Rs^2 \ (1)$$

$$\sum r_b^2 r_c^2 = \left(\sum r_b r_c\right)^2 - 2r_a r_b r_c\left(\sum r_a\right) = s^4 - 2s^2 r(4R+r) \overset{Gerretsen}{\le}$$
$$\le s^2(4R^2+4Rr+3r^2-8Rr-2r^2) = s^2(2R-r)^2 \ (2)$$

$$\sum h_a^3 \overset{Lemma}{\le} \sum (r_b r_c)\sqrt{r_b r_c} \overset{CBS}{\le} \sqrt{\left(\sum (r_b r_c)\right)\left(\sum r_b^2 r_c^2\right)} \overset{(2)}{\le}$$
$$\le \sqrt{s^2 s^2 (2R-r)^2} = s^2(2R-r) \ (3)$$

$$\textit{from } (1)\&(2) \textit{ we need to show } (4R+r)^3 - 12Rs^2 \ge s^2(2R-r) \text{ or}$$
$$(4R+r)^3 \ge s^2(14R-r) \text{or } (4R+r)^3 \overset{Gerretsen}{\ge} (4R^2+4Rr+3r^2)(14R-r) \text{or}$$

$$(4x+1)^3 \overset{\frac{R}{r}=x>2}{\ge} (4x^2+4x+3)(14x-1) \text{ or}$$
$$64x^3+48x^2.+12x+1 \ge 56x^3+52x^2+38x-3 \text{ or}$$
$$4x^3-2x^2-13x+2 \ge 0$$

$$\text{or}(x-2)(4x^2+6x-2) \ge 0 \text{ true as } x \ge 2$$

$$\textit{Equality holds for } a=b=c$$

1894. In ΔABC the following relationship holds:

$$129r^3 - 6R^3 \le \sum r_a^3 \le 82R^3 - 575r^3$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum r_a^3 = \left(\sum r_a\right)^3 - 3\left(\left(\sum r_a\right)\left(\sum r_a r_b\right) - r_a r_b r_c\right) =$$
$$= (4R+r)^3 - 3\left((4R+r)s^2 - s^2 r\right) = (4R+r)^3 - 12Rs^2\ (1)$$

$$Now\ from (1) \sum r_a^3 \overset{Euler\ \&\ Mitrinovic}{\leq} (4R+r)^3 - 12R(16Rr - 5r^2) =$$
$$= (64R^3 + 48R^2 r + 12Rr^2 + r^3) - 12(2r)27r^2 =$$
$$= (64R^3 + 48R^2 r + 12Rr^2 + r^3) - 648r^3$$

$$We\ need\ to\ show\ that:$$

$$(64R^3 + 48R^2 r + 12Rr^2 + r^3) - 648r^3 \leq 82R^3 - 575r^3\ or$$
$$6\,(3R^3 - 8R^2 r - 2Rr^2 + 12r^3) \geq 0\ or$$
$$3x^3 - 8x^2 - 2x + 12 \overset{\frac{R}{r}=x\geq 2}{\geq} 0\ or$$
$$(x-2)(3x^2 - 2x - 6) \geq 0\ or$$
$$(x-2)\left(x(x-2) + 2(x^2-3)\right) \geq 0\ true\ as\ x \geq 2$$
$$From\ (1) \sum r_a^3 \overset{Gerretsen}{\geq} (4R+r)^3 - 12R(4R^2 + 4Rr + 3r^2) =$$
$$= (64R^3 + 48R^2 r + 12Rr^2 + r^3) - 12R(4R^2 + 4Rr + 3r^2)$$

$$We\ need\ to\ show\ that:$$

$$(64R^3 + 48R^2 r + 12Rr^2 + r^3) - 12R(4R^2 + 4Rr + 3r^2) \geq 129r^3 - 6R^3$$

$$or\ 22R^3 - 24Rr^2 - 128r^3 \geq 0\ or$$
$$2(11R^3 - 12Rr^3 - 64r^3) \geq 0\ or$$
$$(R-2r)(11R^2 + 22Rr + 32r^2) \geq 0\ true$$

$$Equality\ holds\ for\ a = b = c$$

1895. In ΔABC the following relationship holds:

$$\sum a\left(\sin\frac{B}{2} + \sin\frac{C}{2}\right) \leq 3\sqrt{3}R$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ \sin\frac{A}{2}\geq\sin\frac{B}{2}\geq\sin\frac{C}{2}\ and$$

$$\sin\frac{A}{2}+\sin\frac{B}{2}\geq\sin\frac{B}{2}+\sin\frac{C}{2}\geq\sin\frac{C}{2}+\sin\frac{A}{2}$$

$$\sum a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\overset{Chebyshev}{\leq}\frac{1}{3}\left(\sum a\right)\left(\sum\sin\frac{A}{2}+\sin\frac{B}{2}\right)=\frac{2s}{3}.2\left(\sum\sin\frac{A}{2}\right)\overset{Jensen}{\leq}$$

$$\leq\frac{4s}{3}3\sin\frac{\pi}{6}=2s\leq 3\sqrt{3}R(\ Mitrinovic)$$

$$Equality\ for\ a=b=c$$

1896. **In ΔABC the following relationship holds:**

$$a^{m_a}b^{m_b}c^{m_c}\leq\left(R\sqrt{3}\right)^{\frac{9R}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ m_a\leq m_b\leq m_c$$

$$am_a+bm_b+cm_c\overset{Chebyshev}{\leq}\frac{1}{3}(a+b+c)(m_a+m_b+m_c)=$$
$$=\frac{1}{3}2s(m_a+m_b+m_c)(1)$$

$$(m_a+m_b+m_c)\overset{Leuenberger}{\leq}4R+r\overset{Euler}{\leq}\frac{9R}{2}\ (2)$$

$$\frac{am_a+bm_b+cm_c}{(m_a+m_b+m_c)}\overset{(1)}{\leq}\frac{\frac{1}{3}2s(m_a+m_b+m_c)}{(m_a+m_b+m_c)}=\frac{1}{3}2s\overset{Mitrinovic}{\leq}\frac{2}{3}\frac{3\sqrt{3}R}{2}=\sqrt{3}R\ (3)$$

$$a^{m_a}b^{m_b}c^{m_c}\overset{AM-GM}{\leq}\left(\frac{am_a+bm_b+cm_c}{(m_a+m_b+m_c)}\right)^{(m_a+m_b+m_c)}\overset{(2)\&(3)}{\leq}\left(R\sqrt{3}\right)^{\frac{9R}{2}}$$

$$Equality\ for\ a=b=c$$

1897. In ΔABC the following relationship holds:

$$m_a r_a + m_b r_b + m_c r_c \le \frac{27R^2}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c\ then\ m_a \le m_b \le m_c\ and\ r_a \ge r_b \ge r_c$$

$$m_a r_a + m_b r_b + m_c r_c \overset{Chebyshev}{\le} \frac{1}{3}\left(\sum m_a\right)\left(\sum r_a\right) \overset{Leuenberger}{\le}$$

$$\le \frac{1}{3}(4R+r)(4R+r) \overset{Euler}{\le} \frac{1}{3}\left(4R+\frac{R}{2}\right)^2 = \frac{27R^2}{4}$$

$$Equality\ for\ a = b = c$$

1898. In ΔABC the following relationship holds:

$$\sum \frac{a}{b+c}\tan\frac{A}{2} \ge \frac{\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c\ \ then \tan\frac{A}{2} \ge \tan\frac{B}{2} \ge \tan\frac{C}{2}\ and \frac{a}{b+c} \ge \frac{b}{c+a} \ge \frac{c}{a+b}$$

$$\sum \frac{a}{b+c}\tan\frac{A}{2} \overset{Chebycv}{\ge} \frac{1}{3}\left(\sum \frac{a}{b+c}\right)\left(\sum \tan\frac{A}{2}\right) \overset{Nesbitt}{\ge} \ge \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{4R+r}{s} \overset{Doucet}{\ge} \frac{\sqrt{3}}{2}$$

$$Equality\ for\ a = b = c$$

1899.

In any acute Δ ABC, the following relationship holds :

$$\left(\sin\frac{A}{2}\right)^{\sin\frac{A}{2}} + \left(\sin\frac{B}{2}\right)^{\sin\frac{B}{2}} + \left(\sin\frac{C}{2}\right)^{\sin\frac{C}{2}} \ge \frac{3\sqrt{2}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} \overset{A-G}{\geq} 3\prod_{cyc}\left(\left(\sin\frac{A}{2}\right)^{\frac{\sin\frac{A}{2}}{3}}\right) \overset{?}{\geq} \frac{3\sqrt{2}}{2} \Leftrightarrow \sum_{cyc}\left(\frac{\sin\frac{A}{2}}{3}.\ln\left(\sin\frac{A}{2}\right)\right) \overset{?}{\geq} \ln\frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \sum_{cyc}\left(\sin\frac{A}{2}.\ln\left(\sin\frac{A}{2}\right)\right) \underset{(*)}{\underbrace{\overset{?}{\geq}}} 3\ln\frac{1}{\sqrt{2}}$$

$$\text{Let } f(x) = \sin\frac{x}{2}.\ln\left(\sin\frac{x}{2}\right)\ \forall x \in \left(0, \frac{\pi}{2}\right) \textit{ and then}:$$

$$f''(x) = \frac{1 - \left(\sin^2\frac{x}{2}\right)\left(2 + \ln\left(\sin\frac{x}{2}\right)\right)}{4\sin\frac{x}{2}} \to (1)$$

$$\because \ln\left(\sin\frac{x}{2}\right) \leq \sin\frac{x}{2} - 1 \therefore \left(\sin^2\frac{x}{2}\right)\left(2 + \ln\left(\sin\frac{x}{2}\right)\right) \leq \left(\sin^2\frac{x}{2}\right)\left(1 + \sin\frac{x}{2}\right) \overset{?}{<} 1$$

$$\Leftrightarrow \csc^2\frac{x}{2} \overset{?}{>} 1 + \sin\frac{x}{2} \Leftrightarrow \cot^2\frac{x}{2} \overset{?}{>} \sin\frac{x}{2} \Leftrightarrow \cos^2\frac{x}{2} \overset{?}{>} \sin^3\frac{x}{2} \to \text{true} \because 0 < \frac{x}{2} < \frac{\pi}{4}$$

$$\Rightarrow \cos^2\frac{x}{2} > \sin^2\frac{x}{2} \overset{1>\sin\frac{x}{2}}{>} \sin^3\frac{x}{2} \therefore \left(\sin^2\frac{x}{2}\right)\left(2 + \ln\left(\sin\frac{x}{2}\right)\right) < 1 \overset{\text{via }(1)}{\Rightarrow} f''(x) > 0$$

$$\therefore \sum_{cyc}\left(\sin\frac{A}{2}.\ln\left(\sin\frac{A}{2}\right)\right) \overset{\text{Jensen}}{\geq} 3\sin\frac{\pi}{6}.\ln\left(\sin\frac{\pi}{6}\right) = \frac{3}{2}.\ln\frac{1}{2} = 3\ln\frac{1}{\sqrt{2}} \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\sin\frac{A}{2}\right)^{\sin\frac{A}{2}} + \left(\sin\frac{B}{2}\right)^{\sin\frac{B}{2}} + \left(\sin\frac{C}{2}\right)^{\sin\frac{C}{2}} \geq \frac{3\sqrt{2}}{2}$$

$$\forall \textit{ acute } \Delta\, ABC, \text{"} = \text{"} \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1900. **In ΔABC the following relationship holds:**

$$\sum m_a \csc A \geq 6\sqrt{3}r$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{1}{h_a} = \frac{1}{r}, \sum h_a \overset{AM-HM}{\geq} \frac{9}{\sum\frac{1}{h_a}} = 9r \ \ (1)$$

$$WLOG\ a \geq b \geq c \text{ then } m_a \leq m_b \leq m_c$$

$$\sum m_a \csc A = 2R\sum\frac{m_a}{a} \overset{Chebyshev}{\geq} 2R \cdot \frac{1}{3}\left(\sum m_a\right)\left(\sum\frac{1}{a}\right) \geq$$

$$\geq\ 2R\frac{1}{3}\left(\sum h_a\right)\left(\sum\frac{1}{a}\right)\overset{Leuenberger\ \&\ (1)}{\geq}\ 2R\frac{1}{3}(9r)\frac{\sqrt{3}}{R}=6\sqrt{3}r$$

$$Equality\ for\ \ a=b=c$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru