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Two Approaches to Formulate Derangement Function

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Abstract

In this article, we intend to formulate the derangement function D_n for $n \geq 2$, by using two approaches, i.e. pure combinatorial approach and abstract algebraic approach through symmetric group. Then, we get two formulas of D_n and find a new interesting identity.

Key words and Phrases: Derangement, combinatorial proof, symmetric group.

1 Introduction

Firstly, we suppose to have n baskets respectively numbered by $1, 2, 3, \ldots, n$ and n balls also respectively numbered by $1, 2, 3, \ldots, n$. Let us call basket k as a basket numbered by k. Call similarly for ball k. In the origin, every basket is filled by one ball in such a way that basket k is filled by ball k. Suppose that we define a **movement** as our way to shift the ball k from basket k to basket $f(k)$ ($\forall k = 1, 2, \ldots, n$) exactly once so that $\{f(1), f(2), \ldots, f(n)\} = \{1, 2, \ldots, n\}$. In other word, f is a bijective mapping from $\{1, 2, \ldots, n\}$ to itself. The number of possible movements so that, in the last position, ball k does not lie inside the basket k (i.e. $f(k) \neq k$, $\forall k = 1, 2, \ldots, n$) is denoted by D_n, formally called as derangement function of n.

D_n is defined as the number of permutations of the n objects in which no object appears in its original position. In abstract algebra, these n distinct objects can be figured as a set of n elements $\{1, 2, \ldots, n\}$. The group of all bijective mappings from $\{1, 2, \ldots, n\}$ to itself with the mapping composition as the group operation is called **symmetric group** S_n. An element of S_n is called a **permutation**. Therefore, D_n is the number of permutations in S_n in which the permutation has no fixed point.

In this article, we emphasize to find the formulas of D_n for $n \geq 2$ (because when $n = 1$, it is trivial that $D_1 = 0$). There are two great formulas of D_n. The first one is

$$D_n = n! \sum_{k=2}^{n} \frac{(-1)^k}{k!}, \ \forall n \geq 2$$

and the second one is

$$D_n = n! \sum_{(x_2, x_3, \ldots, x_n) \in H_n} \frac{1}{2^{x_2} 3^{x_3} \cdots n^{x_n} x_2! x_3! \cdots x_n!}, \ \forall n \geq 2$$

where $H_n = \{(x_2, x_3, \ldots, x_n) \mid 2x_2 + 3x_3 + \cdots + nx_n = n; \ x_2, x_3, \ldots, x_n \in \mathbb{N}_0\}$.
The first formula is obtained by using pure combinatorial approach. While, the second formula is obtained by using abstract algebraic methods through symmetric group concepts.

2 Combinatorial Approach

Consider n baskets (basket 1, basket 2, ..., basket n) and n balls (ball 1, ball 2, ..., ball n) so that ball k lies inside the basket k. We will do a movement once for these balls. We have known that D_n is the number of possible movements so that, in the last position, ball k does not lie inside the basket k again ($\forall k = 1, 2, \ldots, n$).

Suppose that for $k \in \{1, 2, \ldots, n\}$, P_k is defined as the set of all possible movements in such a way that after the movement the ball k still lies inside the basket k, i.e., ball k does not move. According to Inclusion-Exclusion Principle, we have

$$|P_1 \cup P_2 \cup \cdots \cup P_n| = \sum_{k=1}^{n} (-1)^{k+1} \sum |P_1 \cap P_2 \cap \cdots \cap P_k| \quad (1)$$

where

$$\sum |P_1 \cap P_2 \cap \cdots \cap P_k| = \sum_{\substack{1 \leq a_1 < a_2 < \cdots < a_k \leq n; \\ a_1, a_2, \ldots, a_k \in \mathbb{Z}^+}} |P_{a_1} \cap P_{a_2} \cap \cdots \cap P_{a_k}|$$

For every positive integers $a_1, a_2, \ldots, a_k$ with $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, $P_{a_1} \cap P_{a_2} \cap \cdots \cap P_{a_k}$ is the set of movements for which at least ball a_1, ball a_2, ..., ball a_k do not move.

Therefore, $|P_{a_1} \cap P_{a_2} \cap \cdots \cap P_{a_k}| = (n-k)!$ as the number of possibilities the other $n-k$ balls (besides ball a_1, ball a_2, ..., ball a_k) shift their positions. Consequently, for every $k \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}
\sum |P_1 \cap P_2 \cap \cdots \cap P_k| &= \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_k \le n; \\ a_1, a_2, \ldots, a_k \in \mathbb{Z}^+}} |P_{a_1} \cap P_{a_2} \cap \cdots \cap P_{a_k}| \\
&= \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_k \le n; \\ a_1, a_2, \ldots, a_k \in \mathbb{Z}^+}} (n-k)! \\
&= \binom{n}{k}(n-k)! \\
&= \frac{n!}{k!}
\end{aligned}$$

By equation (1), we get

$$\begin{aligned}
|P_1 \cup P_2 \cup \cdots \cup P_n| &= \sum_{k=1}^{n} (-1)^{k+1} \sum |P_1 \cap P_2 \cap \cdots \cap P_k| \\
&= n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}
\end{aligned}$$

Since $P_1 \cup P_2 \cup \cdots \cup P_n$ is the set of all movements for which there is a ball that does not move, so $(P_1 \cup P_2 \cup \cdots \cup P_n)^c$ is the set of all movements so that no ball does not move. Then we obtain

$$\begin{aligned}
D_n &= |(P_1 \cup P_2 \cup \cdots \cup P_n)^c| \\
&= n! - |P_1 \cup P_2 \cup \cdots \cup P_n| \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \quad \forall n \in \mathbb{Z}^+
\end{aligned}$$

and the following theorem follows.

Theorem 2.1. *For all integers $n \ge 2$, we have*

$$D_n = n! \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

3 Symmetric Group Approach

We remember again that the n^{th} symmetric group S_n is the group of all bijective maps from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ with the operation of map composition. Every elements of S_n is called permutation. A permutation f in S_n is called $k-cycle$ (or cycle of length k) iff there exists exactly k distinct numbers $a_1, a_2, \ldots, a_k$ in $\{1, 2, \ldots, n\}$ in such a way that

$$f(a_1) = a_2, \quad f(a_2) = a_3, \quad \cdots\cdots \quad f(a_{k-1}) = a_k, \quad f(a_k) = a_1$$

and $f(x) = x$ for all $x \in \{1, 2, \ldots, n\} \backslash \{a_1, a_2, \ldots, a_k\}$. In this case, we can write the notation $f = (a_1 \ \ a_2 \ \ \cdots \ \ a_{k-1} \ \ a_k)$. Two cycles $(x_1 \ \ x_2 \cdots x_k)$ and $(y_1 \ \ y_2 \cdots y_l)$ are called disjoint iff $\{a_1, a_2, \ldots, a_k\} \cap \{y_1, y_2, \ldots, y_l\} = \emptyset$.
Besides that, when we have $a \in \{1, 2, \ldots, n\}$ and $f \in S_n$, then f fixes a iff $f(a) = a$ and f moves a iff $f(a) \neq a$. If f fixes a, then a is a fixed point of f.

Lemma 3.1. *A permutation can be represented as the product of disjoint cycles. The expression of this product can be called as the disjoint cycles form. The disjoint cycles form of every permutation is unique.*

Example 3.2. *The disjoint cycles form of permutations*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \in S_4, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 2 & 4 \end{pmatrix} \in S_6, \quad \phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 4 & 3 & 5 & 6 & 8 & 9 & 1 \end{pmatrix} \in S_9$$

are $(1 \ \ 3)(2)(4)$, $(1)(2 \ \ 6 \ \ 4 \ \ 3 \ \ 5)$, $(1 \ \ 7 \ \ 8 \ \ 9)(2)(3 \ \ 4)(5)(6)$ *respectively.*

Definition 3.3 (Cycle Pattern)**.** *Cycle pattern of a permutation f in S_n is $1^{x_1}.2^{x_2}.\cdots.n^{x_n}$ iff the disjoint cycles form of f contains x_1 1-cycles, x_2 2-cycles, x_3 3-cycles, …, x_n n-cycles. Moreover, $n = x_1 + 2x_2 + 3x_3 + \cdots + nx_n$ with $x_1, x_2, \ldots, x_n \in \mathbb{N}_0$.*

Example 3.4. *The cycle patterns of permutations*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \in S_4, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 2 & 4 \end{pmatrix} \in S_6, \quad \phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 4 & 3 & 5 & 6 & 8 & 9 & 1 \end{pmatrix} \in S_9$$

are $1^2.2^1$, $1^1.5^1$, *and* $1^3.2^1.4^1$ *respectively.*

Theorem 3.5. *The number of permutations in S_n with cycle pattern $1^{x_1}.2^{x_2}.\cdots.n^{x_n}$ is*

$$\frac{n!}{1^{x_1}2^{x_2}\cdots n^{x_n}x_1!x_2!\cdots x_n!}.$$

Proof. Let $f \in S_n$ has cycle pattern $1^{x_1}.2^{x_2}.\cdots.n^{x_n}$. So, the disjoint cycles form of f contains x_1 1-cycles, x_2 2-cycles, …, x_n n-cycles. We will find the number of such possible f, in S_n.

Let us imagine a k-cycle by a cycle table with having k identic chairs surrounding it. Therefore, we can suppose the disjoint cycles form of f as similar as considering x_k cycle tables each is surrounded by k chairs ($\forall k = 1, 2, \ldots, n$). The total of chairs in overall are $n = x_1 + 2x_2 + \cdots + nx_n$. We let to imagine the numbers $1, 2, \ldots, n$ as n distinct people and these n people will be placed to n provided chairs in such a way that one chair is placed by one person. We intend to design these n chairs identic and these $x_1 + x_2 + \cdots + x_n$ cycle tables identic too. Obviously, the number of $f \in S_n$ (with cycle pattern $1^{x_1}.2^{x_2}.\cdots.n^{x_n}$) is the number of way to position n people in the provided n chairs.

Next, let us consider that there are x_1 cycle tables surrounded by 1 chair, x_2 cycle tables surrounded by 2 chairs, x_3 cycle tables surrounded by 3 chairs, … ,and x_n cycle tables surrounded by n chairs.

If we assume that all tables are different and the permutations of people positions in a one same table is not noticed, so the number of way to positioning these n people to sit down at n chairs is

$$M = \frac{n!}{(1!)^{x_1}(2!)^{x_2}\cdots(n!)^{x_n}}.$$

Suppose we remove the assumption "all tables are different". Let us consider any $k \in \{1, 2, \ldots, n\}$ and consider x_k cycles tables with each of these is surrounded by k people in the chairs. So there are $x_k!$ possibilities to permutating x_k groups of k people cycling x_k tables. Suppose we also remove the assumption "the permutations of people positions in a one same table is not noticed", so the number of permutations of k people cycling a table with k chairs is $(k-1)!$.

By seeing that all the tables are identic and the people (in a same table) can permutate their positions cycling their own tables, the actual number of possibilities to position these n people is

$$M \times \frac{1}{x_1!x_2!\cdots x_n!} \times (0!)^{x_1}(1!)^{x_2}\cdots((n-1)!)^{x_n}$$

$$= \frac{n!}{(1!)^{x_1}(2!)^{x_2}\cdots(n!)^{x_n}} \times \frac{1}{x_1!x_2!\cdots x_n!} \times (0!)^{x_1}(1!)^{x_2}\cdots((n-1)!)^{x_n}$$

$$= \frac{n!}{1^{x_1}2^{x_2}\cdots n^{x_n}x_1!x_2!\cdots x_n!}.$$

In conclusion, $\frac{n!}{1^{x_1}2^{x_2}\cdots n^{x_n}x_1!x_2!\cdots x_n!}$ is the number of permutations in S_n with having cycle pattern $1^{x_1}.2^{x_2}.\cdots.n^{x_n}$. □

Definition 3.6. *Suppose that for every integers $n \geq 2$, we have*

$$H_n = \{(x_2, x_3, \ldots, x_n) \mid 2x_2 + 3x_3 + \cdots + nx_n = n;\ x_2, x_3, \ldots, x_n \in \mathbb{N}_0\}.$$

In symmetric group S_n with $n \geq 2$, a permutation with no fixed point is a permutation whose the product of disjoint cycles does not contain 1-cycle. Equivalently, this permutation has cycle pattern $2^{x_2}.3^{x_3} \cdots .n^{x_n}$ where $2x_2 + 3x_3 + \cdots + nx_n = n$ and $x_2, x_3, \ldots, x_n \in \mathbb{N}_0$. The number of such permutations (in S_n) is

$$\sum_{(x_2,x_3,\ldots,x_n)\in H_n} \frac{n!}{2^{x_2}3^{x_3} \cdots n^{x_n} x_2!x_3! \cdots x_n!}$$

and this expression represents D_n.

Theorem 3.7. *For every integers $n \geq 2$, the number of permutations in S_n which has no fixed point is*

$$D_n = n! \sum_{(x_2,x_3,\ldots,x_n)\in H_n} \frac{1}{2^{x_2}3^{x_3} \cdots n^{x_n} x_2!x_3! \cdots x_n!}.$$

4 Conclusion

By combining Theorem 2.1 and Theorem 3.7, we have the following formulas of derangement function:

$$D_n = n! \sum_{k=2}^{n} \frac{(-1)^k}{k!}, \quad \forall n \geq 2$$

and

$$D_n = n! \sum_{(x_2,x_3,\ldots,x_n)\in H_n} \frac{1}{2^{x_2}3^{x_3} \cdots n^{x_n} x_2!x_3! \cdots x_n!}, \quad \forall n \geq 2$$

where $H_n = \{(x_2, x_3, \ldots, x_n) \mid 2x_2 + 3x_3 + \cdots + nx_n = n;\ x_2, x_3, \ldots, x_n \in \mathbb{N}_0\}$.
These two formulas are equivalent. By comparing them, we get a new identity:

$$\sum_{k=2}^{n} \frac{(-1)^k}{k!} = \sum_{(x_2,x_3,\ldots,x_n)\in H_n} \frac{1}{2^{x_2}3^{x_3} \cdots n^{x_n} x_2!x_3! \cdots x_n!}, \quad \forall n \geq 2 \tag{2}$$

Additional Comment.
Observe that by setting n going to $+\infty$ in equation (2), we get

$$\lim_{n\to\infty} \sum_{(x_2,x_3,\ldots,x_n)\in H_n} \frac{1}{2^{x_2}3^{x_3}\cdots n^{x_n} x_2! x_3! \cdots x_n!} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{(-1)^k}{k!} = \frac{1}{e}$$

where e is Euler number.

References

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