ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 1$ and $\lambda \geq \frac{27}{2}$, then :

$$\sum_{cyc} \frac{x^2}{1-z} + \lambda xyz \leq \frac{1}{2} + \frac{\lambda}{27}$$

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$$\sum_{cyc} \frac{x^2}{1-z} + \lambda xyz \leq \frac{1}{2} + \frac{\lambda}{27} \overset{x+y+z=1}{\Leftrightarrow} \frac{1}{\sum_{cyc} x} \cdot \sum_{cyc} \frac{x^2}{x+y} - \frac{1}{2} \leq \frac{\lambda}{27} - \frac{\lambda xyz}{\left(\sum_{cyc} x\right)^3}$$

$$\Leftrightarrow \frac{1}{\left(\sum_{cyc} x\right) \cdot \prod_{cyc}(x+y)} \cdot \left(\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) + \sum_{cyc} x^2y^2\right) - \frac{1}{2}$$

$$\leq \frac{\lambda}{27\left(\sum_{cyc} x\right)^3} \cdot \left(\left(\sum_{cyc} x\right)^3 - 27xyz\right)$$

$$\Leftrightarrow \frac{\lambda}{27\left(\sum_{cyc} x\right)^2} \cdot \left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) \overset{(*)}{\geq}$$

$$\frac{1}{2\prod_{cyc}(x+y)} \cdot \left(2\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) + 2\sum_{cyc} x^2y^2 - \left(\sum_{cyc} x\right) \cdot \prod_{cyc}(x+y)\right)$$

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0$, $b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a$, $c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s$

$$\Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c \therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and,}$$

$$\sum_{cyc} xy = \sum_{cyc} (s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2 \text{ and,}$$

$$\sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=} s^2 - (4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2 \text{ and,} \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x$$

$$\overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=} (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} r^2((4R+r)^2 - 2s^2)$$

$$\because \lambda \geq \frac{27}{2} \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\frac{s^3 - 27r^2s}{2s^2} \geq \frac{2(4Rr + r^2)(s^2 - 8Rr - 2r^2) + 2r^2((4R+r)^2 - 2s^2) - 4Rrs^2}{8Rrs}$$

$$\big(\text{via } (\bullet), (\bullet\bullet), (\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet\bullet)\big) \Leftrightarrow s^2 + 16R^2 - 46Rr + r^2 \geq 0 \to \text{true}$$

$$\because s^2 + 16R^2 - 46Rr + r^2 \overset{\text{Gerretsen}}{\geq} 16R^2 - 30Rr - 4r^2 = 2(8R+r)(R-2r) \overset{\text{Euler}}{\geq} 0$$

$$\Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{x^2}{1-z} + \lambda xyz \leq \frac{1}{2} + \frac{\lambda}{27}$$

$$\forall\, x, y, z > 0 \,\big|\, x + y + z = 1 \text{ and } \lambda \geq \frac{27}{2}, '' ='' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$