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If $a, \mathrm{b}, \mathrm{c} > 0, ab + \mathrm{bc} + \mathrm{c}a = a + \mathrm{b} + \mathrm{c}$ and $\mathrm{n} \in \mathbb{N}, \lambda \geq 0$, then :

$$\mathrm{n}(a + \mathrm{b} + \mathrm{c}) + \lambda \sum_{cyc} \frac{1}{a^{\mathrm{n}+1}} \geq 3(\mathrm{n} + \lambda)$$

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$$\mathrm{n} \in \mathbb{N} \Rightarrow \mathrm{n} + 1 \geq 1 \therefore \mathrm{n}(a + \mathrm{b} + \mathrm{c}) + \lambda \sum_{cyc} \frac{1}{a^{\mathrm{n}+1}} =$$

$$\mathrm{n}(a + \mathrm{b} + \mathrm{c}) + \lambda \sum_{cyc} \left(1 + \frac{1}{a} - 1\right)^{\mathrm{n}+1} \overset{\text{Bernoulli}}{\geq} \mathrm{n} \sum_{cyc} a + \lambda \sum_{cyc} \left(1 + (\mathrm{n}+1)\left(\frac{1}{a} - 1\right)\right)$$

$$= \mathrm{n} \sum_{cyc} a + 3\lambda + \lambda(\mathrm{n}+1)\left(\sum_{cyc} \frac{1}{a} - 3\right) \overset{?}{\geq} 3(\mathrm{n} + \lambda)$$

$$\Leftrightarrow \mathrm{n}\left(\sum_{cyc} a - 3\right) + \frac{\lambda(\mathrm{n}+1)}{abc}\left(\sum_{cyc} ab - 3abc\right) \overset{?}{\geq} 0$$

$$\overset{ab+bc+ca\,=\,a+b+c}{\Leftrightarrow} \mathrm{n}\left(\sum_{cyc} a - \frac{3\sum_{cyc} ab}{\sum_{cyc} a}\right) + \frac{\lambda(\mathrm{n}+1)}{abc}\left(\frac{\sum_{cyc} ab}{\sum_{cyc} a} . \sum_{cyc} ab - 3abc\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{\mathrm{n}}{\sum_{cyc} a}\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab\right) + \frac{\lambda(\mathrm{n}+1)}{abc\sum_{cyc} a}\left(\left(\sum_{cyc} ab\right)^2 - 3abc\sum_{cyc} a\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because \mathrm{n}, \lambda, \left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab, \left(\sum_{cyc} ab\right)^2 - 3abc\sum_{cyc} a \geq 0$$

$$\therefore \mathrm{n}(a + \mathrm{b} + \mathrm{c}) + \lambda \sum_{cyc} \frac{1}{a^{\mathrm{n}+1}} \geq 3(\mathrm{n} + \lambda)\ \forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\middle|\, ab + \mathrm{bc} + \mathrm{c}a = a + \mathrm{b} + \mathrm{c}$$

and $\mathrm{n} \in \mathbb{N}, \lambda \geq 0,'' ='' $ iff $\mathrm{n} = 0 \wedge a = \mathrm{b} = \mathrm{c} = 1$ (QED)