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If $a,b,c>0,\ a+b+c=3abc$ and $n\in\mathbb{N},\lambda\ge 0$, then :

$$n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}\ge 3(n+\lambda)$$

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$$n\in\mathbb{N}\Rightarrow n+1\ge 1\ \therefore\ n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}$$

$$=n(a+b+c)+\lambda\sum_{cyc}\left(1+\frac{1}{a}-1\right)^{n+1}\overset{Bernoulli}{\ge}$$

$$n\sum_{cyc}a+\lambda\sum_{cyc}\left(1+(n+1)\left(\frac{1}{a}-1\right)\right)$$

$$=n\sum_{cyc}a+3\lambda+\lambda(n+1)\left(\sum_{cyc}\frac{1}{a}-3\right)\overset{?}{\ge}3(n+\lambda)$$

$$\Leftrightarrow n\left(\sum_{cyc}a-3\right)+\frac{\lambda(n+1)}{abc}\left(\sum_{cyc}ab-3abc\right)\overset{?}{\ge}0$$

$$\overset{a+b+c=3abc}{\Leftrightarrow}\ n\left(\sum_{cyc}a-\frac{\sum_{cyc}a}{abc}\right)+\frac{\lambda(n+1)}{abc}\left(\sum_{cyc}ab-3abc\right)\overset{?}{\ge}0$$

$$\Leftrightarrow\frac{n}{abc}\left(\sum_{cyc}a\right)(abc-1)+\frac{\lambda(n+1)}{abc}\left(\sum_{cyc}ab-3abc\right)\boxed{\overset{?}{\underset{(*)}{\ge}}}\,0$$

$$\text{Now, }3abc=\sum_{cyc}a\overset{A-G}{\ge}3\sqrt[3]{abc}\Rightarrow abc\ge 1\rightarrow(1)\text{ and }\sum_{cyc}ab\ge\sqrt{3abc\sum_{cyc}a}$$

$$\overset{a+b+c=3abc}{=}\sqrt{3abc.3abc}\Rightarrow\sum_{cyc}ab\ge 3abc\rightarrow(2)\ \therefore(1)\text{ and }(2)\Rightarrow(*)\text{ is true}$$

$$\therefore n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}\ge 3(n+\lambda)\ \forall\, a,b,c>0\,\middle|\,a+b+c=3abc\text{ and}$$

$$n\in\mathbb{N},\lambda\ge 0,''='' \text{ iff } n=0\wedge a=b=c=1\text{ (QED)}$$