ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xyz \geq 1$ and $\lambda \geq 0$, then :

$$\sum_{cyc} \frac{x^{16} - x^4 + 3}{x^2y^2 + \lambda z(x+y)} \geq \frac{9}{2\lambda + 1}$$

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$$\sum_{cyc} x^2y^2 \geq \frac{1}{3}\left(\sum_{cyc} xy\right)\left(\sum_{cyc} xy\right) \overset{A-G}{\geq} \frac{1}{3}\left(3.\sqrt[3]{x^2y^2z^2}\right)\left(\sum_{cyc} xy\right) \overset{xyz\geq 1}{\geq} \sum_{cyc} xy$$

$$\overset{\because \lambda \geq 0}{\Rightarrow} \lambda \sum_{cyc} xy \leq \lambda \sum_{cyc} x^2y^2 \rightarrow (1)$$

$$\text{Again, } x^{16} - x^4 + 3 = (x^{16} + 1 + 1 + 1) - x^4 \overset{A-G}{\geq} 4x^4 - x^4$$

$$\Rightarrow x^{16} - x^4 + 3 \geq 3x^4 \text{ and analogs} \rightarrow (2)$$

$$\text{We have : } \sum_{cyc} \frac{x^{16} - x^4 + 3}{x^2y^2 + \lambda z(x+y)} \overset{via\ (2)}{\geq} 3\sum_{cyc} \frac{x^4}{x^2y^2 + \lambda z(x+y)} \overset{Bergstrom}{\geq}$$

$$\frac{3\left(\sum_{cyc} x^2\right)^2}{\sum_{cyc} x^2y^2 + 2\lambda \sum_{cyc} xy} \overset{via\ (1)}{\geq} \frac{3\left(\sum_{cyc} x^2\right)^2}{\sum_{cyc} x^2y^2 + 2\lambda \sum_{cyc} x^2y^2} \geq$$

$$\geq \frac{9\sum_{cyc} x^2y^2}{(2\lambda + 1)\sum_{cyc} x^2y^2} = \frac{9}{2\lambda + 1}$$

$$\therefore \sum_{cyc} \frac{x^{16} - x^4 + 3}{x^2y^2 + \lambda z(x+y)} \geq \frac{9}{2\lambda + 1}$$

$$\forall\, x, y, z > 0 \,\big|\, xyz \geq 1 \text{ and } \lambda \geq 0, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$