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If $a, b, c > 0, \lambda \geq 0, ab + bc + ca \leq 3abc$ then:

$$\sum \frac{1}{1 + \lambda a} \leq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$ab + bc + ca \leq 3abc \text{ or, } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 \quad (1)$$

$$\begin{aligned} \sum \frac{1}{1 + \lambda a} &= \sum \left(1 - \left(\frac{\lambda a}{1 + \lambda a} \right) \right) = 3 - \lambda \sum \frac{a}{1 + \lambda a} = \\ &= 3 - \lambda \sum \frac{1}{\frac{1}{a} + \lambda} \stackrel{CBS}{\leq} 3 - \lambda \frac{(1 + 1 + 1)^2}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 3\lambda} \stackrel{(1)}{\leq} 3 - \frac{9\lambda}{3 + 3\lambda} = 3 - \frac{3\lambda}{1 + \lambda} = \frac{3}{\lambda + 1} \end{aligned}$$

Equality holds for $a = b = c = 1$.