ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $xy + yz + zx \leq 3$, then prove that :

$$\frac{2}{\sqrt{xyz}} + \frac{27}{(2x+y)(2y+z)(2z+x)} \geq 3$$

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By AM − GM inequality, we have

$$\frac{2}{\sqrt{xyz}} + \frac{27}{(2x+y)(2y+z)(2z+x)} = \frac{1}{\sqrt{xyz}} + \frac{1}{\sqrt{xyz}} + \frac{27}{(2x+y)(2y+z)(2z+x)} \geq$$

$$\geq 3\sqrt[3]{\left(\frac{1}{\sqrt{xyz}}\right)^2 \cdot \frac{27}{(2x+y)(2y+z)(2z+x)}} = \frac{9}{\sqrt[3]{z(2x+y).x(2y+z).y(2z+x)}} \geq$$

$$\geq \frac{3.9}{z(2x+y) + x(2y+z) + y(2z+x)} = \frac{9}{xy + yz + zx} \geq 3.$$

Equality holds iff $x = y = z = 1$.