ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $xyz = 1$, then prove that :

$$\frac{1}{1+(1+x)^3}+\frac{1}{1+(1+y)^3}+\frac{1}{1+(1+z)^3}\ge\frac{1}{3}$$

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$$\frac{1}{1+(1+x)^3}+\frac{1}{1+(1+y)^3}+\frac{1}{1+(1+z)^3}=$$

$$=\sum_{cyc}\frac{1}{(1+1+x)\left(1+(1+x)^2-(1+x)\right)}=\frac{1}{2}\sum_{cyc}\frac{(2+x)-x}{(x+2)(x^2+x+1)}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{1}{x^2+x+1}-\frac{1}{2}\sum_{cyc}\frac{x}{(x+2)(x^2+x+1)}\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{2}+\frac{\sum_{cyc}\left((y^2+y+1)(z^2+z+1)\right)-\prod_{cyc}(x^2+x+1)}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{2}\sum_{cyc}\frac{x}{(x+2)(3x)}=$$

$$=\frac{1}{2}+\frac{\sum_{cyc}x^2+\sum_{cyc}x+2-xyz-x^2y^2z^2-xyz\sum_{cyc}x-xyz\sum_{cyc}xy}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{6}\sum_{cyc}\frac{1}{x+2}\overset{xyz=1}{=}$$

$$=\frac{1}{2}+\frac{\sum_{cyc}x^2+\sum_{cyc}x+2-1-1-\sum_{cyc}x-\sum_{cyc}xy}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{6}\cdot\frac{\sum_{cyc}xy+4\sum_{cyc}x+12}{9+2\sum_{cyc}xy+4\sum_{cyc}x}=$$

$$=\frac{1}{3}+\frac{1}{6}+\frac{\sum_{cyc}x^2-\sum_{cyc}xy}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{6}\cdot\frac{\sum_{cyc}xy+4\sum_{cyc}x+12}{9+2\sum_{cyc}xy+4\sum_{cyc}x}\ge$$

$$\ge\frac{1}{3}+\frac{1}{6}\left(1-\frac{\sum_{cyc}xy+4\sum_{cyc}x+12}{9+2\sum_{cyc}xy+4\sum_{cyc}x}\right)=$$

$$=\frac{1}{3}+\frac{\sum_{cyc}xy-3}{9+2\sum_{cyc}xy+4\sum_{cyc}x}\ge\frac{1}{3}\left(\because\sum_{cyc}xy\overset{AM-GM}{\ge}3.\sqrt[3]{x^2y^2z^2}\overset{xyz=1}{=}3\right)$$

$$\therefore\frac{1}{1+(1+x)^3}+\frac{1}{1+(1+y)^3}+\frac{1}{1+(1+z)^3}\ge\frac{1}{3}$$

$$\forall\, x,y,z>0\,\middle|\, xyz=1, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$