ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in \mathbb{R}$ and $a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3) \ge 8$,

then prove that : $a^2+b^2+c^2 \ge 3$

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$$\text{Firstly}, \forall\, b, c \in \mathbb{R}, (b^2+c^2)^5 \overset{?}{\ge} 8b^2c^2(b^3+c^3)^2$$
$$\Leftrightarrow \frac{(b^2+c^2)^5}{c^{10}} \overset{?}{\ge} \frac{8b^2c^2(b^3+c^3)^2}{c^{10}} \Leftrightarrow (x^2+1)^5 \overset{?}{\ge} 8x^2(x^3+1)^2 \left(x = \frac{b}{c}\right)$$
$$\Leftrightarrow x^{10} - 3x^8 + 10x^6 - 16x^5 + 10x^4 - 3x^2 + 1 \overset{?}{\ge} 0$$
$$\Leftrightarrow (x-1)^2\left(x^8 + 2x^7 - 2x^5 + 6x^4 - 2x^3 + 2x + 1\right) \overset{?}{\ge} 0$$
$$\Leftrightarrow (x-1)^2\left(4x^4 + \frac{4\left(x^8+2x^7-2x^5+x^4\right)}{4} + \frac{4(x^4-2x^3+2x+1)}{4}\right) \overset{?}{\ge} 0$$
$$\Leftrightarrow (x-1)^2\left(4x^4 + \frac{x^4\left((2x-1)^2\left(\left(x+\frac{3}{2}\right)^2+\frac{1}{2}\right)+\frac{5}{4}\right)}{4} + \frac{(2x+1)^2\left(\left(x-\frac{3}{2}\right)^2+\frac{1}{2}\right)+\frac{5}{4}}{4}\right) \overset{?}{\ge} 0 \to \text{true } \forall\, x \in \mathbb{R}$$
$$\therefore b^2c^2(b^3+c^3)^2 \le \frac{(b^2+c^2)^5}{8} \text{ and analogs } \forall\, a, b, c \in \mathbb{R} \to (1)$$

$$\text{Now}, a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3) \ge 8$$

$$\Rightarrow 64 \le a^4b^4c^4(a^3+b^3)^2(b^3+c^3)^2(c^3+a^3)^2 =$$

$$= \prod_{\text{cyc}}\left(b^2c^2(b^3+c^3)^2\right) \overset{\text{via } (1)}{\le} \prod_{\text{cyc}} \frac{(b^2+c^2)^5}{8}$$

$$\Rightarrow 2^{15} \le \prod_{\text{cyc}}(b^2+c^2)^5 \Rightarrow 8 \le \prod_{\text{cyc}}(b^2+c^2)$$
$$\Rightarrow 2 \le \sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \overset{A-G}{\le} \frac{2(a^2+b^2+c^2)}{3}$$
$$\therefore a^2+b^2+c^2 \ge 3 \;\forall\, a, b, c \in \mathbb{R} \,\middle|\, a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3) \ge 8$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$