ROMANIAN MATHEMATICAL MAGAZINE

If $0 < b < a \leq 4$ and $2ab \leq 3a + 4b$, then prove that :

$$a^2 + b^2 \leq 25$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If $b \leq 3$, then : $a^2 + b^2 \leq 4^2 + 3^2 = 25$ and so, we now focus on the scenario when : $\boxed{b > 3}$ and then : $b > \frac{3}{2} \Rightarrow 2b - 3 > 0 \therefore 2ab \leq 3a + 4b$

$$\Rightarrow a(2b-3) \leq 4b \Rightarrow 0 < b < a \overset{(*)}{\leq} \frac{4b}{2b-3} \Rightarrow 2b - 3 < 4 \Rightarrow b \overset{(**)}{<} \frac{7}{2}$$

$$\text{Now, via } (*),\ a^2 + b^2 \leq \frac{16b^2}{(2b-3)^2} + b^2 \overset{?}{<} 25$$

$$\Leftrightarrow 4b^4 - 12b^3 - 75b^2 + 300b - 225 \overset{?}{<} 0 \Leftrightarrow (b-3)(4b^3 - 75b + 75) \overset{?}{<} 0$$

$$\Leftrightarrow (b-3)\big((b-3)(2b-7)(2b+13) + 26(2b-7) - 16\big) \overset{?}{<} 0 \to \text{true}$$

$$\because (b-3) > 0 \text{ and } (b-3)(2b-7)(2b+13) + 26(2b-7) - 16 < 0 \text{ via } (**)$$

$$\therefore a^2 + b^2 < 25 \text{ and combining all cases, } a^2 + b^2 \leq 25$$

$$\text{for } 0 < b < a \leq 4 \wedge 2ab \leq 3a + 4b, '' ='' \text{ iff } (a = 4, b = 3) \text{ (QED)}$$