ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \in [-2,2]$, then prove that :

$$2(x^6+y^6+z^6)-x^4y^2-y^4z^2-z^4x^2 \le 192$$

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$-2 \le x \le 2 \Rightarrow (x+2)(x-2) \le 0 \Rightarrow x^2-4 \le 0 \Rightarrow 4-x^2 \ge 0$ and similarly, $4-y^2 \ge 0$ and $4-z^2 \ge 0$ and setting $a = 4-x^2, b = 4-y^2$, $c = 4-z^2$, we notice that

: $\boxed{0 \le a, b, c \le 4}$ ($\because a = 4-x^2 \le 4$ and analogs) and $x^2 = 4-a, y^2 = 4-b, z^2 = 4-c$ and via such substitutions,

$$2(x^6+y^6+z^6)-x^4y^2-y^4z^2-z^4x^2 \le 192$$

$$\Leftrightarrow 2\sum_{cyc}(4-a)^3 - \sum_{cyc}\left((4-a)^2(4-b)\right) - 192 \le 0$$

$$\Leftrightarrow \sum_{cyc} a^2b + 20\sum_{cyc} a^2 \overset{(*)}{\le} 2\sum_{cyc} a^3 + 8\sum_{cyc} ab + 48\sum_{cyc} a$$

Now, $(a-4)^2 \ge 0 \Rightarrow a^2+16 \ge 8a \Rightarrow 2a^3+32a \ge 16a^2$ ($\because a \ge 0$) and analogs

$$\Rightarrow 2\sum_{cyc} a^3 + 32\sum_{cyc} a \ge 16\sum_{cyc} a^2 \rightarrow (1)$$

$$\text{Also, } 16\sum_{cyc} a \ge 4\sum_{cyc} a^2$$

$\rightarrow (2)$ ($\because 4 \ge a \Rightarrow 4a \ge a^2$ ($\because a \ge 0$) and analogs) and

moreover, $4\sum_{cyc} ab \ge a^2b+b^2c+c^2a$ ($\because 4 \ge a, b, c$ and $ab, bc, ca \ge 0$)

$$\Rightarrow 8\sum_{cyc} ab \ge \sum_{cyc} a^2b + 4\sum_{cyc} ab \overset{ab,bc,ca \ge 0 \Rightarrow \sum_{cyc} ab \ge 0}{\ge} \sum_{cyc} a^2b \rightarrow (3)$$

$$\therefore (1)+(2)+(3) \Rightarrow 2\sum_{cyc} a^3 + 8\sum_{cyc} ab + 48\sum_{cyc} a \ge \sum_{cyc} a^2b + 20\sum_{cyc} a^2$$

$\Rightarrow (*)$ is true $\therefore 2(x^6+y^6+z^6)-x^4y^2-y^4z^2-z^4x^2 \le 192 \ \forall x, y, z \in [-2,2]$,

$'' ='' $ $(x=2, y=2, z=2)$ or $(x=-2, y=-2, z=-2)$

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