ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in \mathbb{R}$ *and* $abc(16a^4+1)(16b^4+1)(16c^4+1) \geq 4913$,

then prove that :

$$a^2+b^2+c^2 \geq 3$$

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$$abc(16a^4+1)(16b^4+1)(16c^4+1) \geq 4913 = 17^3$$

$$\Rightarrow a^2b^2c^2(16a^4+1)^2(16b^4+1)^2(16c^4+1)^2 \geq 17^6$$

$$\Rightarrow \sqrt[3]{a^2b^2c^2(16a^4+1)^2(16b^4+1)^2(16c^4+1)^2} \geq 289$$
$$\Rightarrow \sqrt[3]{xyz(16x^2+1)^2(16y^2+1)^2(16z^2+1)^2} \geq 289 \; (x = a^2, y = b^2, z = c^2)$$
$$\Rightarrow \ln\left(\left(\sqrt[3]{x(16x^2+1)^2}\right)\left(\sqrt[3]{y(16y^2+1)^2}\right)\left(\sqrt[3]{z(16z^2+1)^2}\right)\right) \geq \ln 289$$
$$\Rightarrow \sum_{\text{cyc}} \ln \sqrt[3]{x(16x^2+1)^2} \geq \ln 289 \; (\because a, b, c \neq 0 \Rightarrow x, y, z > 0)$$

$$\Rightarrow \sum_{\text{cyc}} \ln(x(16x^2+1)^2) \overset{(*)}{\geq} 3\ln 289$$

$$\text{Now, } f(x) = \ln(x(16x^2+1)^2) \text{ is concave as } f''(x) = -\frac{1280x^4 - 32x^2 + 1}{x^2(16x^2+1)^2} < 0$$

$$\left(\begin{matrix}\because \text{discriminant of } (1280x^4 - 32x^2 + 1) = 1024 - 5120 < 0 \\ \Rightarrow 1280x^4 - 32x^2 + 1 > 0\end{matrix}\right)$$
$$\therefore \sum_{\text{cyc}} \ln(x(16x^2+1)^2) \overset{\text{Jensen}}{\leq} 3\ln(t(16t^2+1)^2) \left(t = \frac{1}{3}\sum_{\text{cyc}} x > 0\right)$$

$$\overset{\text{via }(*)}{\Rightarrow} 3\ln(t(16t^2+1)^2) \geq 3\ln 289 \Rightarrow t(256t^4 + 32t^2 + 1) \geq 289 \Rightarrow$$
$$256t^5 + 32t^3 + t - 289 \geq 0 \Rightarrow (t-1)(256t^4 + 256t^3 + 288t^2 + 288t + 289) \geq 0$$

$$\Rightarrow t = \frac{1}{3}\sum_{\text{cyc}} x \geq 1 \Rightarrow x + y + z \geq 3 \therefore a^2 + b^2 + c^2 \geq 3$$
$$\forall a, b, c \in \mathbb{R} \,\big|\, abc(16a^4+1)(16b^4+1)(16c^4+1) \geq 4913,$$

$$'' ='' \text{ iff } (a = b = c = 1) \text{ or } (a = 1, b = c = -1) \text{ or } (a = b = -1, c = 1)$$

$$\text{or } (a = -1, b = 1, c = -1) \text{ (QED)}$$