ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $3xyz \geq x + y + z$, then prove that :

$$\frac{xy + yz + zx - 1}{\sqrt{3x^2+1} + \sqrt{3y^2+1} + \sqrt{3z^2+1}} \geq \frac{1}{3}$$

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$$3xyz \geq x + y + z \Rightarrow \left(\sum_{cyc} xy\right)^2 \geq 3xyz\left(\sum_{cyc} x\right) \geq \left(\sum_{cyc} x\right)^2 \geq 3\sum_{cyc} xy$$

$$\therefore \sum_{cyc} xy \geq \sum_{cyc} x \rightarrow (1) \text{ and } \sum_{cyc} xy \geq 3 \rightarrow (2)$$

$$\text{Now, } 3xyz \geq x + y + z \Rightarrow 1 \leq \frac{3xyz}{\sum_{cyc} x} \Rightarrow \sqrt{3x^2+1} \leq \sqrt{3x^2 + \frac{3xyz}{\sum_{cyc} x}}$$

$$\therefore \sqrt{3x^2+1} \leq \sqrt{\frac{3}{\sum_{cyc} x}}.\sqrt{x}.\sqrt{x^2 + \sum_{cyc} xy} \text{ and analogs}$$

$$\Rightarrow \sqrt{3x^2+1} + \sqrt{3y^2+1} + \sqrt{3z^2+1} \overset{CBS}{\leq} \sqrt{\frac{3}{\sum_{cyc} x}}.\sqrt{\sum_{cyc} x}.\sqrt{\sum_{cyc} x^2 + 3\sum_{cyc} xy}$$

$$= \sqrt{3}.\sqrt{\left(\sum_{cyc} x\right)^2 + \sum_{cyc} xy} \overset{via\ (1)}{\leq} \sqrt{3}.\sqrt{\left(\sum_{cyc} xy\right)^2 + \sum_{cyc} xy}$$

$$\Rightarrow \frac{xy + yz + zx - 1}{\sqrt{3x^2+1} + \sqrt{3y^2+1} + \sqrt{3z^2+1}} \geq \frac{\sum_{cyc} xy - 1}{\sqrt{3}.\sqrt{\left(\sum_{cyc} xy\right)^2 + \sum_{cyc} xy}}$$

$$\left(\because \text{via } (2), \sum_{cyc} xy \geq 3 > 1 \Rightarrow xy + yz + zx - 1 > 0\right) \overset{?}{\geq} \frac{1}{3}$$

$$\Leftrightarrow \frac{t^2 - 2t + 1}{3(t^2 + t)} \overset{?}{\geq} \frac{1}{9} \left(t = \sum_{cyc} xy\right) \Leftrightarrow 2t^2 - 7t + 3 \overset{?}{\geq} 0 \Leftrightarrow (2t-1)(t-3) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t = \sum_{cyc} xy \geq 3, \text{via } (2) \therefore \frac{xy + yz + zx - 1}{\sqrt{3x^2+1} + \sqrt{3y^2+1} + \sqrt{3z^2+1}} \geq \frac{1}{3}$$

$$\forall\, x, y, z > 0 \,\middle|\, 3xyz \geq x + y + z, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$