ROMANIAN MATHEMATICAL MAGAZINE

If $a \geq -1, b \geq -1$ and $a^2 + b^2 \geq 5$, then prove that :

$$a^3 + b^3 \geq 7$$

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$$a \geq -1, b \geq -1 \Rightarrow (a+1)(b+1) \geq 0 \Rightarrow a + b + ab + 1 \geq 0$$
$$\Rightarrow ab \geq -a - b - 1 \therefore a^2 + b^2 \geq 5 \Rightarrow (a+b)^2 \geq 5 + 2ab \geq 5 - 2a - 2b - 2$$
$$\Rightarrow (a+b)^2 + 2(a+b) - 3 \geq 0 \Rightarrow (a+b-1)(a+b+3) \geq 0 \Rightarrow \boxed{a + b \geq 1}$$
$$\rightarrow (1)\ (\because a \geq -1, b \geq -1 \Rightarrow a + b \geq -2 \Rightarrow a + b \nleq -3)$$
$$\boxed{\text{Case 1}}\ b \geq 2 \text{ and via } (1), a \geq 1 - b \Rightarrow a^3 \geq (1-b)^3 = 1 - 3b + 3b^2 - b^3$$
$$\Rightarrow a^3 + b^3 - 7 \geq 3b^2 - 3b - 6 = 3(b-2)(b+1) \overset{b \geq 2}{\geq} 0 \Rightarrow a^3 + b^3 \geq 7$$
$$\boxed{\text{Case 2}}\ b < 2 \text{ and } \because a^2 + b^2 \geq 5 \therefore 4 > b^2 \geq 5 - a^2 \Rightarrow a^2 > 1 \Rightarrow \boxed{a > 1}$$
$$(\because a \geq -1 \Rightarrow a \nless -1)$$
$$\text{Now}, a^2 + b^2 \geq 5 \Rightarrow a^2 \geq 5 - b^2 \Rightarrow a \geq \sqrt{5 - b^2}$$
$$(\because a > 1 > 0 \text{ and } 5 - b^2 > 4 - b^2 > 0) \Rightarrow a^3 \geq (5 - b^2)^{\frac{3}{2}}$$
$$\Rightarrow \boxed{a^3 + b^3 \geq (5 - b^2)^{\frac{3}{2}} + b^3 \overset{?}{>} 7} \Leftrightarrow (5 - b^2)^{\frac{3}{2}} \overset{?}{>} 7 - b^3,$$
which is trivially true if $b \geq \sqrt[3]{7}$ and so, we now focus on the scenario when :
$$b < \sqrt[3]{7} \text{ and then} : (5 - b^2)^{\frac{3}{2}} \overset{?}{>} 7 - b^3 \Leftrightarrow (5 - b^2)^3 \overset{?}{>} (7 - b^3)^2$$
$$\Leftrightarrow 8(2b^6 - 15b^4 - 14b^3 + 75b^2 - 76) \overset{?}{<} 0$$
$$\Leftrightarrow (b-2)(b+1)((2b-3)^2(4(b+2)^2 + 5) + 44(2-b) + 27) \overset{?}{<} 0 \rightarrow \text{true} \because$$
$$1 - \leq b < 2 \therefore a^3 + b^3 > 7 \text{ and combining both cases}, a^3 + b^3 \geq 7$$
$$\forall a \geq -1, b \geq -1 \mid a^2 + b^2 \geq 5,$$
$$'' ='' \text{ iff } (a = 2, b = -1) \text{ or } (a = -1, b = 2)\ (\text{QED})$$