ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ and $x^2+y^2-2z^2+2xy+yz+zx\le 0$, then prove that

$$\frac{x^4+y^4}{z^4}+\frac{y^4+z^4}{x^4}+\frac{z^4+x^4}{y^4}\ge\frac{273}{8}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $x^2+y^2-2z^2+2xy+yz+zx=(x+y-z)(x+y+2z)$, then $z\ge x+y$.

$$\frac{x^4+y^4}{z^4}+\frac{y^4+z^4}{x^4}+\frac{z^4+x^4}{y^4}=\frac{x^4+y^4}{z^4}+\frac{z^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)+\frac{255z^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)+\frac{y^4}{x^4}+\frac{x^4}{y^4}$$

$$\overset{AM-GM}{\ge}2\sqrt{\frac{x^4+y^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)}+\frac{255(x+y)^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)+2$$

$$\overset{AM-GM}{\ge}2\sqrt{\frac{2x^2y^2}{256}\cdot\frac{2}{x^2y^2}}+\frac{255\left(2\sqrt{xy}\right)^4}{256}\cdot\frac{2}{x^2y^2}+2=\frac{273}{8}.$$

Equality holds iff $x=y=\frac{z}{2}$.