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If $a,b,c>0$ and $ab+bc+ca\le 3abc$ then prove that:

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3\le\sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sqrt{2}.\sum_{cyc}\sqrt{b+c}=\sum_{cyc}\sqrt{(1+1)\left(\frac{b^2+c^2}{b+c}+\frac{2bc}{b+c}\right)}\ge\sum_{cyc}\left(\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{2bc}{b+c}}\right)$$

$$=\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+\sum_{cyc}\sqrt{\frac{2}{\frac{1}{b}+\frac{1}{c}}}\overset{Jensen}{\ge}\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+3\sqrt{\frac{2.3}{\sum_{cyc}\left(\frac{1}{b}+\frac{1}{c}\right)}}$$

$$=\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+3\sqrt{\frac{3abc}{ab+bc+ca}}\ge\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+3,$$

as desired. Equality holds iff $a=b=c=1$.