ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $ab(a^4 + b^4) \geq 2$, then prove that :

$$a^5 + b^5 \geq 2$$

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$$\text{We shall prove that}: \left(a^5 + b^5\right)^6 \geq 2a^5b^5(a^4 + b^4)^5 \to (1)$$

$$\text{and} \because a^5 + b^5 \overset{\text{Chebyshev}}{\geq} \frac{1}{2}(a + b)(a^4 + b^4)$$

$$\therefore \left(a^5 + b^5\right)^6 \geq \frac{1}{32}\left(a^5 + b^5\right)(a + b)^5(a^4 + b^4)^5$$

$$\overset{\text{Holder}}{\geq} \frac{1}{32} \cdot \frac{1}{16} \cdot (a + b)^5(a + b)^5(a^4 + b^4)^5 \overset{?}{\geq} 2a^5b^5(a^4 + b^4)^5$$

$$\Leftrightarrow (a + b)^{10} \overset{?}{\geq} 1024a^5b^5 \Leftrightarrow (a + b)^2 \overset{?}{\geq} 4ab \to \text{true via } A - G \therefore (1) \text{ is true}$$

$$\Rightarrow \left(a^5 + b^5\right)^6 \overset{ab(a^4+b^4) \geq 2}{\geq} 64 \Rightarrow a^5 + b^5 \geq 2$$

$$\forall\, a, b > 0 \,\middle|\, ab(a^4 + b^4) \geq 2, '' ='' \text{ iff } a = b = 1 \text{ (QED)}$$