ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $ab = 4$, then prove that :

$$\frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{b^3+1}} \ge \frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = \frac{1}{a}, y = \frac{1}{b} \text{ and then : } xy = \frac{1}{4} \text{ and } t = x + y \overset{A-G}{\ge} 2\sqrt{xy} = 2.\sqrt{\frac{1}{4}}$$

$$\Rightarrow t \ge 1 \to (1) \text{ and now, } \frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{b^3+1}} = \frac{1}{\sqrt{\frac{1}{x^3}+1}} + \frac{1}{\sqrt{\frac{1}{y^3}+1}}$$

$$= \frac{x.\sqrt{x}.\sqrt{x}}{\sqrt{x(x+1)}.\sqrt{x^2-x+1}} + \frac{y.\sqrt{y}.\sqrt{y}}{\sqrt{y(y+1)}.\sqrt{y^2-y+1}} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{(x+y)^2}{\sqrt{x^2+y^2+x+y}.\sqrt{x^2+y^2-(x+y)+2}} = \frac{t^2}{\sqrt{t^2-2xy+t}.\sqrt{t^2-2xy-t+2}}$$

$$\overset{xy=\frac{1}{4}}{=} \frac{2t^2}{\sqrt{(2t^2+2t-1)(2t^2-2t+3)}} \overset{?}{\ge} \frac{2}{3} \Leftrightarrow 5t^4 - 8t + 3 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-1)(5t^3+5t^2+5(t-1)+2) \overset{?}{\ge} 0 \to \text{true} \because t \ge 1 \text{ via } (1)$$

$$\therefore \frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{b^3+1}} \ge \frac{2}{3} \ \forall\, a, b > 0 \,\big|\, ab = 4, '' ='' \text{ iff } a = b = 2 \text{ (QED)}$$