ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $ab(a^4 + a^2b^2 + b^4) \geq 3$, then prove that :

$$a^2 + b^2 \geq 2$$

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$$\text{It's clear that } ab > 0 \text{ and so, } t = \frac{a}{b} = \frac{ab}{b^2} > 0 \text{ and now,}$$
$$\text{we shall prove that : } 3(a^2 + b^2)^3 \geq 8ab(a^4 + a^2b^2 + b^4)$$

$$\Leftrightarrow 3(t^2 + 1)^3 \geq 8t(t^4 + t^2 + 1) \Leftrightarrow 3t^6 - 8t^5 + 9t^4 - 8t^3 + 9t^2 - 8t + 3 \geq 0$$

$$\Leftrightarrow \frac{1}{16}(t - 1)^2\left((12t^2 + 4t + 9)(2t - 1)^2 + 39\right) \geq 0 \rightarrow \text{true}$$

$$\therefore 3(a^2 + b^2)^3 \geq 8ab(a^4 + a^2b^2 + b^4) \geq 24 \Rightarrow (a^2 + b^2)^3 \geq 8$$

$$\Rightarrow a^2 + b^2 \geq 2, \text{"="} \text{ iff } a = b = 1 \text{ or } a = b = -1 \text{ (QED)}$$